Coherence, correlation functions, and spectral width

When we talked about coherence and its role in interference, we said it largely boiled down to how well correlated waves were with each other (or themselves).

To measure correlation mathematically, you basically slide one function along another function and measure the extent to which they overlap as a function of offset, like so:

Correlation between f_1 and f_2, f_c:

$$f_c(\tau) = \int_{-\infty}^{\infty} f_1^*(t) f_2(t+\tau)\,dt$$

We multiply two functions together and integrate to get the degree to which they overlap, and let that be a function of some offset τ.

Typically, $f_c(0)$ is when you have the most overlap, and as τ increases, $f_c(\tau)$ decreases. Watching how fast that decrease occurs will let you get a quantifiable coherence length.

In that particular case (finding coherence lengths), often times we're interested in how well a function correlates with itself. The so-called autocorrelation function is:

$$f_a(\tau) = \int_{-\infty}^{\infty} f^*(t) f(t+\tau)\,dt$$

In either of these, you'll note a complex conjugate, which makes finding the overlap of complex-valued functions a little more sensible. (It generally kills any oscillation in t in the integrand)

Sometimes, as when you have a function that's infinite in extent, this autocorrelation integral can diverge unless you include some averaging behavior:

$$f_a(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f^*(t) f(t+\tau)\,dt$$

Depending on where you look, you'll see one or the other definition, usually reflecting what kind of function they plan to be working with.

Let's try taking an autocorrelation of a specific function. We'll try a gaussian pulse, so:

$$E(t) = E_0 e^{-t^2/T^2} e^{-i\omega t}$$

Where T is basically the pulse width.

We have:

$$f_a(\tau) = \int_{-\infty}^{\infty} E_0 e^{-t^2/T^2} e^{i\omega t} \cdot E_0 e^{-(t+\tau)^2/T^2} e^{-i\omega(t+\tau)} dt$$

There's a t, T, and τ in here, so be careful. I'll try to make them distinct.

$$f_a(\tau) = E_0^2 e^{i\omega\tau} \int_{-\infty}^{\infty} e^{-(t^2+(t-\tau)^2)/T^2} dt$$

$$= \text{"} \int_{-\infty}^{\infty} e^{-(t^2+t^2-2t\tau+\tau^2)/T^2} dt$$

$$= E_0^2 e^{i\omega\tau} e^{-\tau^2/T^2} \int_{-\infty}^{\infty} e^{-2(t^2-t\tau)/T^2} dt$$

That integral is a standard item in tables, and I get:

$$\int_{-\infty}^{\infty} e^{-2(t^2-t\tau)/T^2} dt = \sqrt{\frac{\pi}{(2/T^2)}}\, e^{\left(\frac{2\tau}{T^2}\right)^2/\left(4\cdot\frac{2}{T^2}\right)}$$

$$= T\sqrt{\pi/2}\, e^{\frac{4\tau^2}{T^4}\cdot\frac{T^2}{8}}$$

$$= T\sqrt{\pi/2}\, e^{\frac{\tau^2}{2T^2}}$$

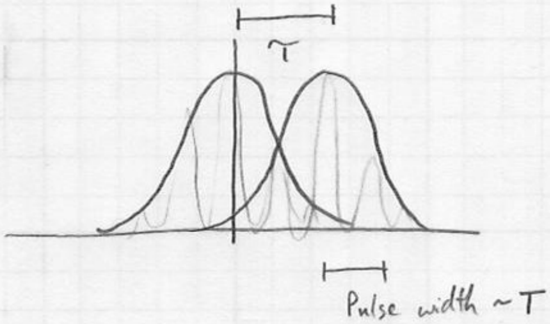
So the whole thing is:

$$f_a(\tau) = E_0^2 e^{i\omega\tau} e^{-\tau^2/T^2} \cdot e^{\tau^2/2T^2} = \boxed{E_0^2 e^{i\omega\tau} e^{-\tau^2/2T^2}}$$

So pretty much another gaussian, with similar width, and now a function of that offset τ.

How do we interpret this?

Take a gaussian pulse, copy it, and offset the copy by some amount τ:



$f_a(\tau)$ is basically measuring the overlap vs. τ. And we see that we get the max value at $\tau = 0$, and that $f_a(\tau)$ drops to zero as τ increases, with the pulse width T setting the length scale.

In interference terms, if you try to interfere a gaussian with another identical gaussian, it'll work fine and you'll get brightness and darkness as you slide the one relative to the other, at least until the offset exceeds T, at which point the interference fringes fade out.

Another thing you'll see is the normalized version of this autocorrelation function, sometimes called the "complex degree of temporal coherence". But it's just a normalized autocorrelation function:

$$f_{a,n}(\tau) \equiv f_a(\tau) / f_a(0)$$

With our Gaussian, that'd yield:

$$f_{a,n}(\tau) = \frac{E_0^2 e^{i\omega\tau} e^{-\tau^2/2T^2}}{E_0^2 \cdot 1} = e^{i\omega\tau} e^{-\tau^2/2T^2}$$

Which has all the relevant info. No one really cares about the absolute amplitude of a correlation function.

Taking this and integrating its square magnitude does a lovely job of quantifying how long it takes for a function to fall out of synch with itself, and is in fact one way to formally define the coherence time for a known function:

$$\tau_c \equiv \int_{-\infty}^{\infty} |f_{a,n}(\tau)|^2 \, d\tau$$

Using that with our gaussian gives:

$$\tau_c = \int_{-\infty}^{\infty} e^{-\tau^2/2T^2} \cdot e^{-\tau^2/2T^2} \, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\tau^2/T^2} \, d\tau = \sqrt{\pi}\, T$$

So the coherence time is within a constant of the pulse width, which is reasonable physically. The coherence length, therefore, would be:

$$l_c = c \cdot \tau_c = \sqrt{\pi}\, cT$$

Which, in concrete terms, tells you roughly by how far you can vary the length of one leg of an interferometer before you lose the interference fringes.

Now, I'm going to give you one more awesome piece of physics before we wrap this up. Monochromatic sine waves tend to have really long coherence lengths. This is strongly related to the broader Fourier-analysis result that a function's temporal width is generally inversely related to a function's spectral width: Monochromatic (spectrally narrow) waves are temporally wide, and short pulses are spectrally wide.

Something that you can prove generally is that the spectral density of some signal, $S(\omega)$, is the Fourier transform of the autocorrelation function:

$$S(\omega) = \int_{-\infty}^{\infty} f_a(\tau)\, e^{i\omega\tau} \, d\tau$$

$S(\omega)$ tells you how much intensity a signal has at any particular frequency; it has units of Watts/m^2 per Hz, and if you integrate $S(\omega)$ over all ω you get I.

Proving the above would get us kind of off track, but if you really want to see it, look up the Wiener-Khinchin theorem.

If you take <u>that</u> to be true, a bit more deriving tells you something <u>really</u> interesting: that the coherence time for some signal is related to the spectral width quite simply:

$$\Delta\omega = \frac{2\pi}{\tau_c}$$

$\Delta\omega$ is the effective width of some signal's frequency domain representation, and when it is small, the coherence length is large, and vice versa. This is consistent with the notion of short pulses (with their necessarily short coherence lengths) having wide spectra, and monochromatic signals ($\Delta\omega \to 0$) having long coherence lengths. It all fits together in a nice, tidy package.