4_9_08

Note Title 4/9/2008

So far:

stationary states of Hydrogen:

3 quantum numbers

n determines the energy

$E_n = E_1 / n^2$

$$E_1 = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]$$

l, m related to orbital angular momentum

classically $\vec{L} = \vec{r} \times \vec{p}$

$$(\vec{L})_i = \epsilon_{ijk} r_j p_k$$

ϵ_{ijk} cyclic permutation symbol

$\epsilon_{ijk} = 0$ if any indices are repeated

$= 1$ if ijk is an even permutation of 123

$= -1$ if ijk is an odd permutation

e.g. $\epsilon_{123} = 1$

$\epsilon_{213} = -1$ $\quad 1 \leftrightarrow 2$ $\quad$ 1 perm.

$\epsilon_{231} = 1$ $\quad 3 \leftrightarrow 1$ $\quad$ 1 more perm

= 2 total

$\epsilon_{221} = 0$ etc.

So using the Einstein summation convention (repeated indices on same side of = are summed, we have

$(\vec{L})_i = \epsilon_{ijk} r_j P_k$ } sum over j + k

$$L_1 = \epsilon_{123} r_2 P_3 + \epsilon_{132} r_3 P_2$$
$$= r_2 P_3 - r_3 P_2 \quad \text{or}$$

$$\boxed{L_x \quad y P_z - z P_y}$$

similarly $L_y = z P_x - x P_z$

$$L_z = x P_y - y P_x$$

The QM observables in coordinate representation:

$$P_x \to -i\hbar \frac{\partial}{\partial x} \quad P_y \to -i\hbar \frac{\partial}{\partial y} \quad \text{etc.}$$

So

$L_x \qquad y P_z - z P_y$

$\Rightarrow \qquad y\left(-i\hbar \frac{\partial}{\partial z}\right) - z\left(-i\hbar \frac{\partial}{\partial y}\right)$

$L_x = \quad -i\hbar\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right]$

$\vdots \qquad \vec{L} = \vec{r} \times (-i\hbar\nabla)$

$[L_x, L_y] = ?$

So $\quad L_x L_y - L_y L_x$

$L_x = y P_z - z P_y$

$L_y = z P_x - x P_z$

$L_z = x P_y - y P_x$

idea: reduce to lots of

$[r_i, p_j] = -[p_i, r_j] = i\hbar\delta_{ij}$

$[r_i, r_j] = [p_i, p_j] = 0$

$= (yP_z - zP_y)(zP_x - xP_z) - (zP_x - xP_z)(yP_z - zP_y)$

$= yP_z\, zP_x - yP_z\, xP_z - zP_y\, zP_x + zP_y\, xP_z$

$- [zP_x\, yP_z - zP_x\, zP_y - xP_z\, yP_z + xP_z\, zP_y$

$\underset{1}{[yP_z, zP_x]} + \underset{2}{[xP_z, yP_z]} + \underset{3}{[zP_x, zP_y]} - \underset{4}{[xP_z, zP_y]}$

Key point: compare terms 2 + 1

2: $xP_z\, yP_z - yP_z\, xP_z$ these operators all commute

$= xy\, P_zP_z - yx\, P_zP_z$

$= 0$

1: $yP_z\, zP_x - zP_x\, yP_z$

$= \underbrace{P_z z}\, yP_x - \underbrace{zP_z}\, yP_x$

cannot exchang their order

$= yP_x [P_z, z]$

Similarly term 3 drops out and term 4:

4: $[zP_y, xP_z]$

$= zP_y\, xP_z - xP_z\, zP_y$

$= xP_y (zP_z - P_z z) = -xP_y [P_z, z]$

So

$[L_x, L_y] = yP_x \underbrace{[P_z, z]}_{-i\hbar} - xP_y \underbrace{[P_z, z]}_{-i\hbar}$

$= i\hbar (xP_y - yP_x) = i\hbar L_z$

$$[L_x, L_y] = i\hbar L_z$$

By the same argument

$$[L_y, L_z] = i\hbar L_x$$

$$[L_z, L_x] = i\hbar L_y$$

Fundamental angular momentum commutation relations

Observation L_x, L_y, L_z are incompatible observables. By our generalized uncertainty principle

$$\sigma_{L_x}^2 \sigma_{L_y}^2 \geq \left(\frac{1}{2i}\langle i\hbar L_z\rangle\right)^2$$

$$= \frac{\hbar^2}{4}\langle L_z\rangle^2$$

Lo and behold (let $L^2 = L_x^2 + L_y^2 + L_z^2$)

$$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0$$

proof on page 161.

Since L^2 commutes with each component we can say

$$[L^2, \vec{L}] = 0$$

Since L^2 is compatible (commutes) with each component of L, lets pick one and solve the simultaneous Σ-value problem:

$$\underbrace{L^2 f = \lambda f \qquad L_z f = \mu f}_{}$$

we can only do this for observables that commute.

we'll do this by the algebraic method. i.e ladder operators.

$$L_\pm = L_x \pm i L_y$$

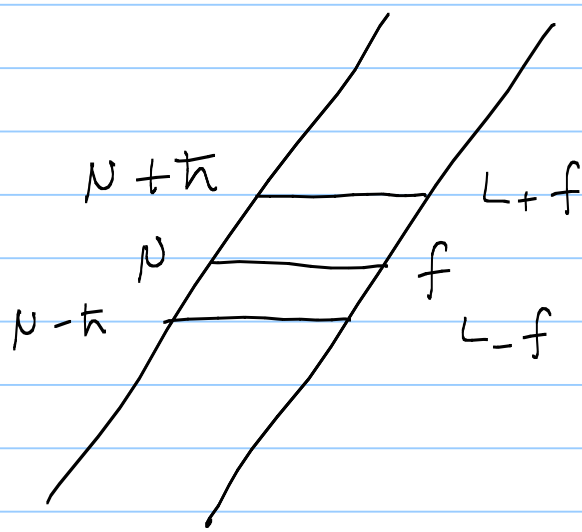
you can show that

$$[L_z, L_\pm] = \pm \hbar L_\pm$$

$$[L^2, L_\pm] = 0$$

we can show that

$$L_z(L_\pm f) = (\mu \pm \hbar) L_\pm f$$



eventually this must terminate or else we'll reach a state for which L_z exceeds $|L|$ which cannot happen. Hence

there exists a state f_t such that

$$L_+ f_t = 0$$

So $\quad L_z f_t = \hbar l f_t \quad$ for some l

show that $\boxed{L^2 f_t = \hbar^2 l(l+1) f_t}$

Similarly by applying L_- we must reach a bottom state

$$L_- f_b = 0 \quad \text{for some } \bar{\ell}$$

so $$L_z f_b = \hbar \bar{\ell} f_b$$

Can show $$L^2 f_b = \hbar^2 \bar{\ell}(\bar{\ell}-1) f_b$$

next time we will show that

$$\bar{\ell}(\bar{\ell}-1) = \ell(\ell+1)$$

$$\Rightarrow \bar{\ell} = -\ell$$

And the E-values of L_z are $m\hbar$ where $-\ell \le m \le \ell$ in integer steps.

$$L^2 f_\ell^m = \hbar^2 \ell(\ell+1) f_\ell^m$$

$$L_z f_\ell^m = \hbar m f_\ell^m$$

$\ell = 0, \frac{1}{2}, 1, \ldots \qquad m = -\ell, -\ell+1, \ldots \ell$

in spherical coord.

$$L_z = -i\hbar \frac{\partial}{\partial \phi}$$

$$L^2 = -\hbar^2 \underbrace{\left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]}_{\text{none other than the angular part of } \nabla^2}$$