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Note Title 4/28/2008

Bosons / Fermions

Consider a system that consists of 2 particles a + b.

Ignore spin for the moment so we can write the joint wavefunction as $\psi(\vec{r}_1, \vec{r}_2)$

In some cases we can write this as the product of 1-particle wavefunctions

$$\psi(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1)\,\psi_b(\vec{r}_2)$$

Not true if particles are "entangled"

This seems reasonable, but I've introduced an error. By writing as I have I've implicitly assumed the particles were distinguishable. In QM there are particles that are fundamentally indistinguishable!

QM accomodates this in the following way:

Suppose $\psi(\vec{r}_1,\vec{r}_2) = A[\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) - \psi_b(\vec{r}_1)\psi_a(\vec{r}_2)]$

Then, exchanging the 2 particles results in

$$A[\psi_b(\vec{r}_1)\psi_a(\vec{r}_2) - \psi_a(\vec{r}_1)\psi_b(\vec{r}_2)]$$

$$= -\psi(\vec{r}_1,\vec{r}_2)$$

But this π phase change has no effect on the amplitude $|\psi(r_1,r_2)|^2$

On the other hand if

$$\psi(\vec{r}_1,\vec{r}_2) = A[\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) + \psi_b(\vec{r}_1)\psi_a(\vec{r}_2)]$$

Then exchanging particles has no effect at all

Does it matter?
Suppose particles a and b are actually identicle:

$$\psi(\vec{r}_1,\vec{r}_2) = A[\psi_a(\vec{r}_1)\psi_a(\vec{r}_2) - \psi_a(\vec{r}_1)\psi_a(\vec{r}_2)] \equiv 0$$

So, particles which generate a π phase factor on exchange cannot be in the same state. Fermions.

Particles which do not generate the phase change have no such limitation. Such particles are Bosons.

We can even define an exchange operator

$$P f(\vec{r}_1, \vec{r}_2) = f(\vec{r}_2, \vec{r}_1)$$

$P^2 = I$. So the eigenvalues of P^2 must be ± 1.

If 2 particles are identical, the Hamiltonian must treat them the same. Hence

$$[P, H] = 0$$

So there are simultaneous eigenstates of H and P.

Fundamental law of nature

$$\boxed{\psi(\vec{r}_1, \vec{r}_2) = \pm\, \psi(\vec{r}_2, \vec{r}_1)}$$

for all solutions of Schrödingers eqn.

Example. infinite square well with 2 noninteracting particles

single-particle states

$$\psi_n(x) = \sqrt{\frac{2}{a}}\, \sin\left(\frac{n\pi}{a} x\right) \qquad E_n = n^2 K$$

$$\left(K = \frac{\pi^2 \hbar^2}{2ma^2}\right)$$

product states

$$\psi_{n_1 n_2}(x_1, x_2) = \psi_{n_1}(x_1)\, \psi_{n_2}(x_2)$$

$$E_{n_1, n_2} = (n_1^2 + n_2^2) K$$

(Because they are noninteracting)

Suppose the 2 particles are identical Bosons, then the G.S. is

$$\psi_{11} = \frac{2}{a} \sin(\pi x_1/a) \sin(\pi x_2/a) \qquad E_{11} = 2K$$

But for fermions $\psi_{11} = 0$, so the GS is

$$\frac{\sqrt{2}}{a} \left[\sin(\pi x_1/a) \sin(2\pi x_2/a) - \sin(2\pi x_1/a) \sin(\pi x_2/a) \right]$$

$$E_{12} = E_{21} = 5K$$