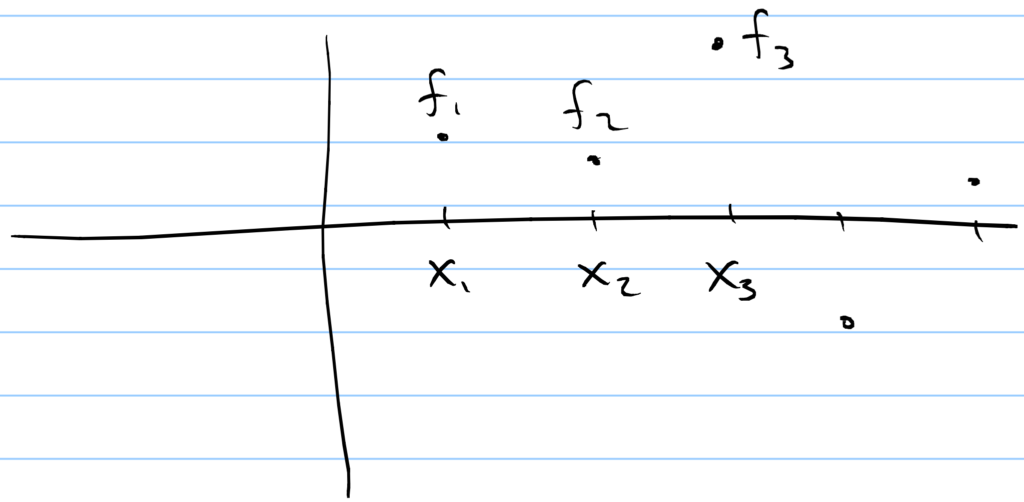
Note Title 11/6/2006

11/6/06

Trig. interpolation of data:

$$(f_k, x_k)$$



$$P(x) = \sum_{n=0}^{N-1} c_n e^{inx}$$

polynomial in e^{ix}

interpolation means

$$\boxed{P(x_k) = f_k}$$

$p(x_k)$

Hence

$$f_k = \sum_{n=0}^{N-1} c_n e^{i n x_k}$$

we said $x_k = \dfrac{2\pi k}{N}$

so $$f_k = \sum_{n=0}^{N-1} c_n \underbrace{e^{i 2\pi n k / N}}_{\text{matrix}_{(k,n)}}$$

call this matrix Q

$$\left(Q^{*T} Q\right)_{k\ell} = \sum_{j=0}^{N-1} \left(Q^{*T}\right)_{kj} Q_{j\ell}$$

$$= \sum_{j=0}^{N-1} e^{-i 2\pi k j / N} e^{i 2\pi j \ell / N}$$

$$= \sum_{j=0}^{N-1} e^{i 2\pi j (\ell - k)/N}$$

$$= \begin{cases} 0 & \text{if } l \neq k \\ \sum_{j=0}^{N-1} 1 & \text{if } l = k \end{cases}$$

$$\sum_{j=0}^{N-1} 1 = N$$

So if we normalize Q by $\frac{1}{\sqrt{N}}$ then

$$Q^{*T} Q = \mathbb{I}$$

So $\vec{f} = Q \vec{c}$

$$\Rightarrow \vec{c} = Q^{*T} \vec{f}$$

$$= Q^{\dagger} \vec{f} \quad \text{DFT}$$

$Q^\dagger$ = Hermitian of Q

$$\boxed{\vec{c} = Q^\dagger \vec{f}}$$

Discrete Fourier Transform examples

$$\vec{c} = Q^\dagger \vec{f}$$

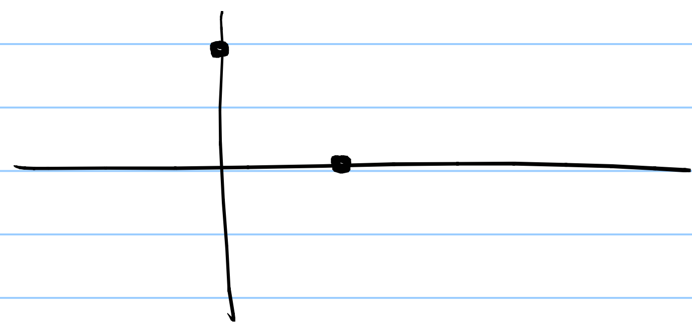
$$c_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n e^{-2\pi i n k/N}$$

You could easily program this.

Algorithm

1) look over all k from $0 \to N-1$

2) For each k do the sum

E.g. 

$f_0 = 1 \qquad f_1 = 0$

$$c_0 = \frac{1}{\sqrt{2}}\left[1 \cdot e^{-0} + 0\right]$$

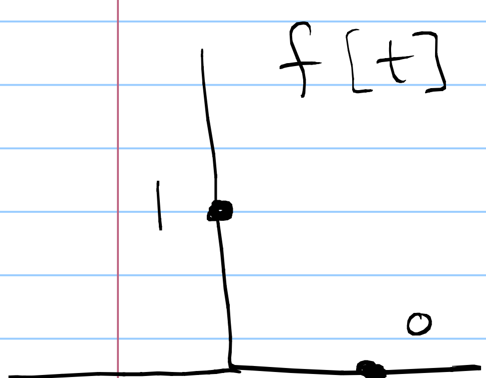
$$c_1 = \frac{1}{\sqrt{2}}\left[1 \cdot e^{-0} + 0\right]$$

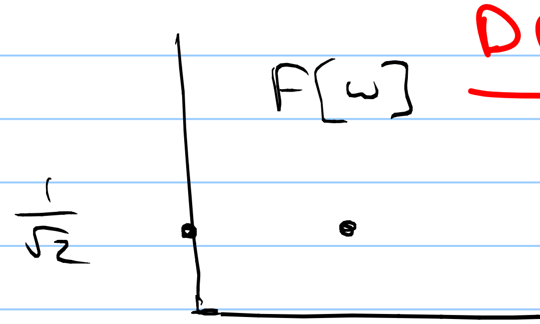
$$c_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n e^{-2\pi i n k/N}$$

So the DFT of the "time series" $\{1, 0\}$ is

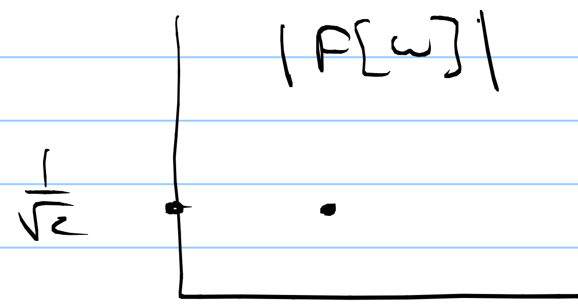
$$\left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}$$

This is purely real so the periodogram is equal to the DFT





DFT



periodogram

Ex. $f = \{0, 1, 0\}$

$$C_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n e^{-2\pi i n k/N}$$

$$c_0 = \frac{1}{\sqrt{3}} \left[1 \quad e^{-2\pi i 1 \cdot 0/N} \right] = .577$$

$$c_1 = \frac{1}{\sqrt{3}} \left[1 \; e^{-2\pi i/3} \right]$$

$$Re[c_1] = \frac{\cos(2\pi/3)}{\sqrt{3}} = -.288$$

$$Im[c_1] = \frac{\sin(2\pi/3)}{\sqrt{3}} =$$

$$c_2 = \frac{1}{\sqrt{3}} \left[1 \cdot e^{-2\pi i 2/3} \right]$$

$$Re[c_2] = \frac{\cos(4\pi/3)}{\sqrt{3}} = -.288$$

$$Im[c_2] = \frac{\sin(4\pi/3)}{\sqrt{3}} = -.5$$

$$\vec{f} = \{0, \quad 1, \quad 0\}$$

$$\vec{c} = \{\underline{.577}, -\underline{.288} + \underline{.5i}, -\underline{.288} - \underline{.5i}\}$$

— im

— re

periodogram

$$|\vec{c}| = \{.577, .577, .577\}$$

The periodogram of a spike is flat No matter where in time the spike is located.

But the DFT of a spike is only flat if the spike is a t=0.

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mathematic

$X = \{1, 1, 1, 1\}$

y= Fourier[x]

Periodogram= Abs[y]

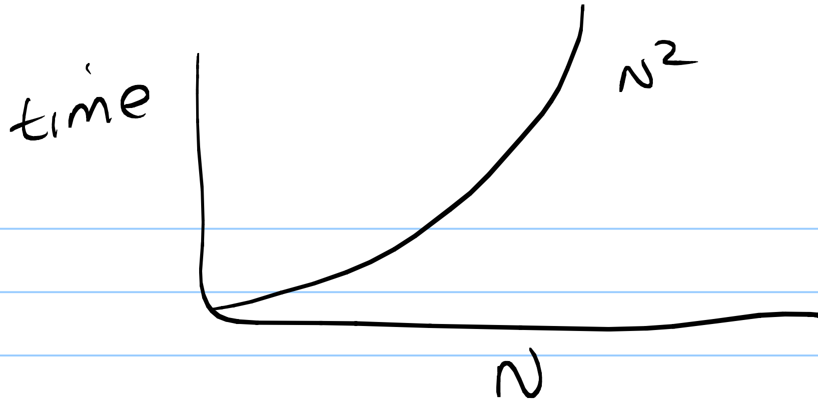
ListPlot [Re[y], plotjoined -> true]

Listplot [Abs[y], ...]

$$C_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n e^{-2\pi i n k/N}$$

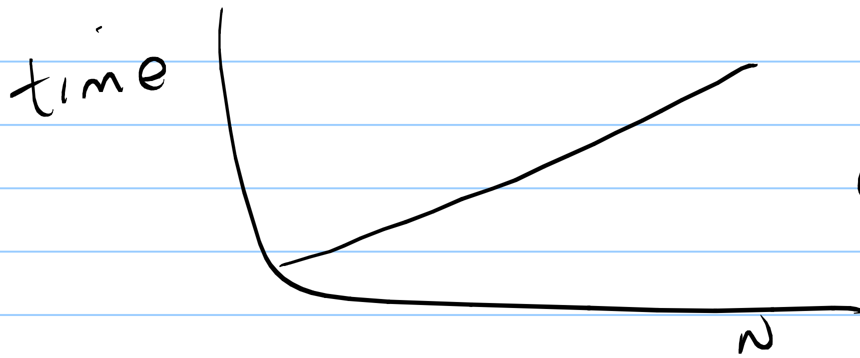
floating point operation

flop $O(N^2)$



DFT

$O(N^2)$



FFT

$O(N \log N)$

Gilbert Strang