Day 26: B + H + Boundary conditions

Let's briefly review how bound charge led to the idea of the displacement field $\vec{D}$.

The source equation $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ refers to <u>all</u> charges, both free and bound

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_f + \rho_b}{\varepsilon_0}$$

This is inconvenient since we often can only specify the free charge. Since $\rho_b = -\vec{\nabla}\cdot\vec{P}$, this invites some rearrangement:

$$\varepsilon_0 \vec{\nabla}\cdot\vec{E} = \rho_f + \rho_b = \rho_f - \vec{\nabla}\cdot\vec{P} \Rightarrow \vec{\nabla}\cdot(\varepsilon_0\vec{E}+\vec{P}) = \rho_f$$

So if we define a composite field $\vec{D} \equiv \varepsilon_0\vec{E}+\vec{P}$ we can recover a source equation in the style of Gauss's law, but written only in terms of free charge:

$$\vec{\nabla}\cdot\vec{D} = \rho_f$$

We're going to do something analogous with magnetism. Amperes law says $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$. That $\vec{J}$ is <u>all</u> $\vec{J}$, free and bound.

$$\vec{\nabla}\times\vec{B} = \mu_0(\vec{J}_f + \vec{J}_b)$$

Since $\vec{J}_b = \vec{\nabla}\times\vec{M}$, we can do some reshuffling:

$$\nabla\times \vec{B}/\mu_0 = \vec{J}_f + \vec{\nabla}\times\vec{M}$$

$$\Rightarrow \quad \nabla\times(\vec{B}/\mu_0 - \vec{M}) = \vec{J}_f \qquad \text{Define } \vec{H} = \vec{B}/\mu_0 - \vec{M}$$

and we get $\boxed{\vec{\nabla}\times\vec{H} = \vec{J}_f}$, Ampere's law in matter

Interlude: Naming $\vec{H}$ + $\vec{B}$ (or, "Grumpy Old People Having a Fight about Stupid Bullshit")

So $\vec{E}$ is fundamental in that it is responsible for forces $(\vec{F} = q\vec{E})$ and out of $\vec{E}$ + $\vec{D}$ is the one that exists in vacuum. Everyone thus calls $\vec{E}$ the electric field and refers to the composite object $\vec{D}$ by a different name, the displacement field

Similarly, $\vec{B}$ is fundamental in that it makes forces $(\vec{F} = q\vec{v}\times\vec{B})$ and out of $\vec{B}$ + $\vec{H}$ is the one that exists in vacuum. So obviously everyone calls $\vec{B}$ the magnetic field and $\vec{H}$ something else, right?

Wrong. For some reason that I cannot find (and I've looked), some people call $\vec{B}$ the magnetic induction and $\vec{H}$ the magnetic field. And other people do other stuff

Names I've heard for:

$\underline{B}$	$\underline{H}$
Magnetic field	Magnetic field
Magnetic induction	Auxiliary magnetic field
Magnetic flux density	Magnetic field strength

And people will fight to the death over this sort of thing!

We're going to keep it clean. $\vec{B}$ is the magnetic field. Period. And $\vec{H}$ we'll just call H.

Anyway, back to Ampere's law in matter: $\nabla \times H = J_f$
or $\oint H \cdot dl = I_{f,enc}$

Let's continue to parallel what was done for polarized dielectrics.

There, $\vec{P} = \varepsilon_0 \chi_e \vec{E}$ for linear, isotropic materials.

and $\vec{D} = \varepsilon \vec{E}$ with $\varepsilon = \varepsilon_0(1+\chi_e)$

Here, once again for linear isotropic materials, we define

$\vec{M} = \chi_m \vec{H}$ (χ_m is the magnetic susceptibility)

$\vec{B} = \mu \vec{H}$ with $\mu = \mu_0(1+\chi_m)$ μ is named the permeability

χ_m is very small for almost all materials save ferromagnetic ones ($\chi_m \sim 10^{-3} - 10^{-8}$)

Thus for <u>most</u> materials magnetization is much less relevant than polarization.

Also, we can derive an important constraint on $\vec{J}_b$ now:

Since $\vec{J}_b = \vec{\nabla} \times \vec{M}$, if $\vec{M} = \chi_m \vec{H}$ then

$\vec{J}_b = \vec{\nabla} \times \chi_m \vec{H}$. And $\vec{\nabla} \times \vec{H} = \vec{J}_f$, so

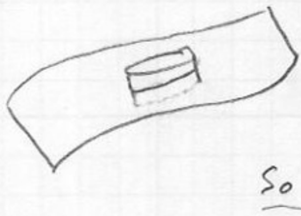
$$\boxed{\vec{J}_b = \chi_m \vec{J}_f}$$

Therefore if our material is linear, uniform, & isotropic, $\vec{J}_b$ can exist only if $\vec{J}_f$ exists.

Boundary conditions on B + H

We get these in the same way we got the conditions on E + D.

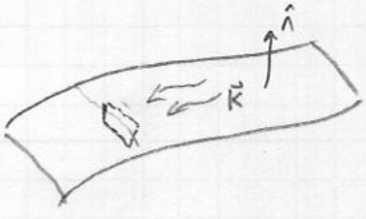
$\vec{\nabla}\cdot\vec{B}=0$ is equivalent to $\oint \vec{B}\cdot d\vec{A}=0$, so if we draw a small box around a surface and look at the flux through that box, only the flux through the top + bottom matter if we make the box thin enough. And so



$$\oint \vec{B}\cdot d\vec{A} = B_{\perp_{above}}A - B_{\perp_{below}}A = 0 \Rightarrow \boxed{B_{\perp\,above} = B_{\perp\,below}}$$

So the normal component of $\vec{B}$ is continuous across surfaces

$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ is equivalent to $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$



Draw an Amperian loop enclosing a surface and make it very thin. Crank out what we've done before

Then $B_{\parallel above}\cancel{L} - B_{\parallel below}\cancel{L} = \mu_0 K\cdot\cancel{L}$

$\Rightarrow \quad B_{\parallel above} - B_{\parallel below} = \mu_0 K$

But we actually have to be a bit more careful this time, because only surface currents that pass through the loop count, which is to say K's that are $\perp$ to the loop face and thus to the $B_{\parallel}$'s being considered. Orientation matters. We respect this by writing

$\boxed{\vec{B}_{\parallel_1} - \vec{B}_{\parallel_2} = \mu_0\vec{K}\times\hat{n}}$ where $\hat{n}$ is the normal vector to the surface. We're coming at this slightly bass-ackwards. Instead of trying to write the component of $\vec{K}$ that goes through the loop, we're looking at the component of $\vec{K}$ that when crossed with $\hat{n}$ is parallel to the B's under consideration. That's the component that matters

For $\vec{H}$, it's much the same:

$\vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot(\vec{B}/\mu_0 - \vec{M}) = 0$ as long as $\vec{\nabla}\cdot\vec{M}=0$. This is usually true (it may be possible to rig an exception with ferromagnetic materials).

So usually $\vec{\nabla}\cdot\vec{H}=0 \Rightarrow \boxed{H_{\perp above} - H_{\perp below} = 0}$

And $\vec{\nabla}\times\vec{H} = \vec{J}_f \Rightarrow \boxed{\vec{H}_{\parallel_1} - \vec{H}_{\parallel_2} = K_f\times\hat{n}}$

Field boundary conditions (in statics) summarized

For E: $E_{\perp_1} - E_{\perp_2} = \sigma/\epsilon_0$, $E_{\parallel}$ is continuous

For B: $B_{\perp}$ is continuous, $\vec{B}_{\parallel_1} - \vec{B}_{\parallel_2} = \mu_0 \vec{K} \times \hat{n}$

For D: $D_{\perp_1} - D_{\perp_2} = \sigma_f$, $D_{\parallel}$ is continuous if $\vec{\nabla} \times \vec{P} = 0$

For H: $H_{\perp}$ is continuous if $\vec{\nabla} \cdot \vec{M} = 0$, $\vec{H}_{\parallel_1} - \vec{H}_{\parallel_2} = \vec{K}_f \times \hat{n}$

You could memorize all these and it wouldn't be a waste of time, but they all come from nearly identical derivations (one of two, anyway):

$\vec{\nabla} \cdot (\vec{\text{field}})$ equations lead to conditions on $\text{field}_{\perp}$

$\vec{\nabla} \times (\vec{\text{field}})$ equations lead to conditions on $\text{field}_{\parallel}$

This is a mathematical and totally general result.