

## Chapter 2

# Waves and Modes in One and Two Spatial Dimensions

There are a number of different techniques for solving the 1-D wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (2.0.1)$$

Perhaps the oldest is the method of traveling waves. In this method we look for solutions of the form  $u(x, t) = f(x + ct)$  and  $u(x, t) = f(x - ct)$ . Using the chain rule you can see that

$$\frac{\partial u}{\partial t} = \pm c f'$$

where the prime denotes differentiation with respect to the argument ( $x + ct$  for instance). Similarly

$$\frac{\partial^2 u}{\partial t^2} = c^2 f''$$

$$\frac{\partial^2 u}{\partial x^2} = f''.$$

As a result, **any** differentiable function evaluated at  $x \pm ct$  is a solution of the 1-D wave equation. Think of the function  $f$  as representing some shape. As time increases  $x$  must increase at the rate  $ct$  in order for the shape to have the same value. This means that the shape, evaluated at  $x - ct$ , is actually moving to the right at a constant speed of  $c$ . Similarly  $x + ct$  is moving to the left at speed  $ct$ .

## 2.1 1-D Separation of Variables: Summary of the Argument

Another approach to solving linear PDE's (not just the wave equation) is separation of variables. In certain coordinate systems we can find solutions which are factored. This means that the multivariate solution can be written as the product of univariate functions. The wave equation is separable in many coordinate systems, including Cartesian, spherical, and cylindrical.

Here is an overview of the argument for one spatial variable in Cartesian coordinates.

We want to solve

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (2.1.1)$$

such that, for example,  $u(0, t) = u(l, t) = 0$  (clamped ends) and  $u(x, 0) = u_0(x)$  and  $\partial u / \partial t(x, 0) = v_0(x)$  where  $u_0$  and  $v_0$  represent the initial displacement and velocity.

Guess a solution of the form  $u(x, t) = X(x)T(t)$ . This doesn't always work. Plug this into Equation 2.1.1 and divide by  $XT$ .

This gives

$$c^2 \frac{X''}{X} = \frac{\ddot{T}}{T}. \quad (2.1.2)$$

This is an equation involving only  $x$  on the left and  $t$  on the right. The only way this can be true is if both sides are constant. Call this constant  $-\omega^2$ .

So  $u = XT$  reduces (2.1.1) to two ODE's:

$$\ddot{T} + \omega^2 T = 0 \quad (2.1.3)$$

and

$$\ddot{X} + \frac{\omega^2}{c^2} X = 0. \quad (2.1.4)$$

Solve these as usual:

$$T(t) = A \cos(\omega t) + B \sin(\omega t) \quad (2.1.5)$$

$$X(x) = C \cos\left(\frac{\omega}{c}x\right) + D \sin\left(\frac{\omega}{c}x\right) \quad (2.1.6)$$

with  $A, B, C, D$  arbitrary constants.

The clamped end boundary conditions imply that  $X(0) = X(l) = 0$ . Therefore  $C = 0$  and  $\omega/c = \pi n/l$ . Leaving:

$$T(t) = A \cos\left(\frac{\pi n c}{l} t\right) + B \sin\left(\frac{\pi n c}{l} t\right) \quad (2.1.7)$$

$$X(x) = D \sin\left(\frac{\pi n}{l} x\right) \quad (2.1.8)$$

Or,

$$u(x, t) = DA \sin\left(\frac{\pi n}{l} x\right) \cos\left(\frac{\pi n c}{l} t\right) + DB \sin\left(\frac{\pi n}{l} x\right) \sin\left(\frac{\pi n c}{l} t\right). \quad (2.1.9)$$

Let's relabel the constant  $DA$ , calling it  $A$ , and  $DB$ , calling it  $B$ . Then

$$u(x, t) = A \sin\left(\frac{\pi n}{l} x\right) \cos\left(\frac{\pi n c}{l} t\right) + B \sin\left(\frac{\pi n}{l} x\right) \sin\left(\frac{\pi n c}{l} t\right). \quad (2.1.10)$$

This solution obviously will not satisfy general initial conditions. However, linearity of the wave equation guarantees that if

$$A \sin\left(\frac{\pi n}{l} x\right) \cos\left(\frac{\pi n c}{l} t\right) + B \sin\left(\frac{\pi n}{l} x\right) \sin\left(\frac{\pi n c}{l} t\right) \quad (2.1.11)$$

is a solution, then so is

$$\sum_n A_n \sin\left(\frac{\pi n}{l} x\right) \cos\left(\frac{\pi n c}{l} t\right) + B_n \sin\left(\frac{\pi n}{l} x\right) \sin\left(\frac{\pi n c}{l} t\right) \quad (2.1.12)$$

where  $A_n$  and  $B_n$  are arbitrary constants.

Now we have some hope of satisfying the initial conditions. Let's see. If

$$u(x, t) = \sum_n A_n \sin\left(\frac{\pi n}{l} x\right) \cos\left(\frac{\pi n c}{l} t\right) + B_n \sin\left(\frac{\pi n}{l} x\right) \sin\left(\frac{\pi n c}{l} t\right) \quad (2.1.13)$$

then

$$u(x, 0) = \sum_n A_n \sin\left(\frac{\pi n}{l} x\right) \quad (2.1.14)$$

and

$$\frac{\partial u(x, t)}{\partial t} = \sum_n B_n \frac{\pi n c}{l} \sin\left(\frac{\pi n}{l} x\right). \quad (2.1.15)$$

So this scheme will work if and only if we can choose the constants  $A_n$  and  $B_n$  such that

$$u_0(x) = \sum_n A_n \sin\left(\frac{\pi n}{l} x\right) \quad (2.1.16)$$

and

$$v_0(x) = \sum_n B_n \frac{\pi n c}{l} \sin\left(\frac{\pi n}{l} x\right). \quad (2.1.17)$$

This is our first example of a Fourier series. We will explore this last conjecture in detail. That this should be possible is not remotely obvious in my opinion and that it is true is one of the great triumphs of 19th century mathematical physics.

What's the simplest solution we could construct? We would displace the string into one of its normal modes, initially at rest, and then let go. This corresponds to making all the  $B_n$  coefficients in 2.1.13 equal to zero (since the initial velocity is zero) and letting all but one of the  $A_n$  be zero. For instance, if we displace the string into its fundamental mode ( $\sin(n\pi x/l)$ , for  $n = 1$ ) then the complete solution is

$$u(x, t) = \sin(\pi x/l) \cos(\pi c t/l). \quad (2.1.18)$$

That's it. Notice that if you start the system out in one of its normal modes it stays there forever. In a linear system there is absolutely no way to transfer energy amongst the modes. Later on we'll be able to prove this directly: the energy of each mode is constant, so whatever energy a particular mode starts out with, it stays that way forever. (This is not too hard to prove. Why don't you give it a try. Just compute the energy (kinetic + potential) and integrate over one complete period of the motion.)



The symbol to the left indicates that on the WWW page you will find a *Mathematica* notebook; in this case one that solves the 1D problem for initial conditions corresponding to the string being pulled up in the middle and released at  $t = 0$ . We use *Mathematica's* built-in Fourier series capability to represent a "hat" function as a 6 term sine-series. (Don't worry about the details of the Fourier analysis, we'll be covering that later.) But download this notebook and run it. You'll see a beautiful and realistic animation.

The results of running this code are shown in Figure 2.1.

## 2.2 2-D separation of variables

Separation of variables for the 2-D wave equation proceeds in the same way.

$$c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial^2 u}{\partial t^2} \quad (2.2.1)$$

We assume a solution of the form

$$u(x, y, t) = X(x)Y(y)T(t). \quad (2.2.2)$$

Equation 2.2.1 then becomes (after dividing by  $XYT$ )

$$c^2 \left( \frac{X''}{X} + \frac{Y''}{Y} \right) = \frac{\ddot{T}}{T}. \quad (2.2.3)$$

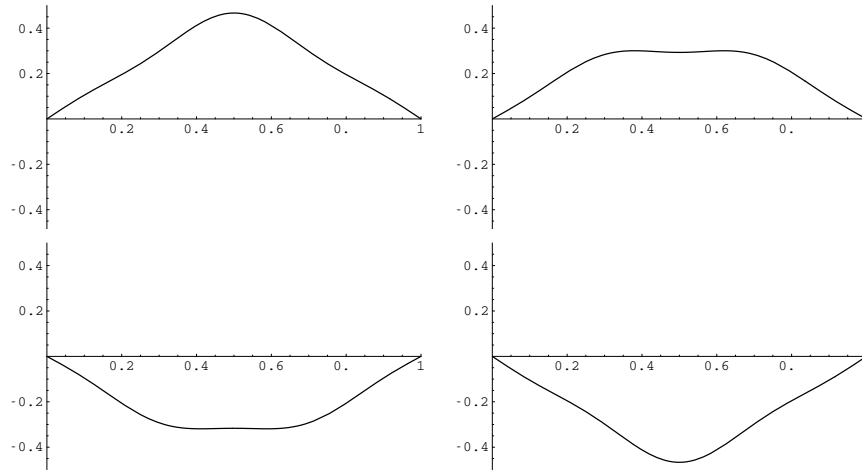


Figure 2.1: 4 of the 50 time snapshots of the plucked string. To compute this I approximated a triangular initial displacement by a 6 term Fourier series. This series is slowly convergent since the triangle function is not differentiable in the middle. But 6 terms are good enough to get a feel for what's happening.

As in 1-D, for this to be true, both sides of this equation must be constant. Let's call this constant  $-\omega^2$ .<sup>1</sup> So we have

$$\frac{\ddot{T}}{T} + \omega^2 = 0 \quad (2.2.4)$$

and

$$c^2 \left( \frac{X''}{X} + \frac{Y''}{Y} \right) = -\omega^2. \quad (2.2.5)$$

Let's rewrite this last equation as

$$\frac{X''}{X} + \frac{\omega^2}{c^2} = -\frac{Y''}{Y}. \quad (2.2.6)$$

We can apply the standard separation of variables argument again: an equation of  $x$  on the left and an equation of  $y$  on the right; this must mean that both sides equal yet another constant. Let's call this one  $k_y^2$  (for reasons that will become apparent shortly):

$$\frac{X''}{X} + \frac{\omega^2}{c^2} = -\frac{Y''}{Y} = k_y^2. \quad (2.2.7)$$

So we have two de-coupled ODE's for the spatial variables

$$X'' + \left( \frac{\omega^2}{c^2} - k_y^2 \right) X = 0 \quad (2.2.8)$$

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<sup>1</sup>You should convince yourself that it doesn't matter what we call this constant, plus, minus, with or without the square. It all works out the same in the end.

and

$$Y'' + k_y^2 Y = 0. \quad (2.2.9)$$

We can preserve the symmetry of these two equations by inventing a new label for  $c^2 - k_y^2$ . We'll call this  $k_x^2$ . Then we have the nice pair of equations

$$X'' + k_x^2 X = 0 \quad (2.2.10)$$

$$Y'' + k_y^2 Y = 0 \quad (2.2.11)$$

where, because of how we've defined  $k_x$  we have

$$\frac{\omega^2}{c^2} = k_x^2 + k_y^2. \quad (2.2.12)$$

The constants  $k_x$  and  $k_y$  have the dimensions of reciprocal length.  $\frac{\omega}{c}$  is one over the wavelength, times  $2\pi$ .

So we've successfully reduced the 2-D wave equation, which is a partial differential equation in two space variables and time, to three un-coupled ODE's. We already know how to solve these equations, so let's consider an interesting particular case. Let's consider a rectangular drum (a thin membrane, clamped on the sides) of lengths  $L_x$  and  $L_y$ . We'll put the origin of the coordinate system at  $x = 0$ ,  $y = 0$ . Then in order for the displacement to be zero at  $x = L_x$  and  $y = L_y$ , we must have

$$X(x) = A \sin(k_x x) \quad (2.2.13)$$

$$Y(y) = B \sin(k_y y) \quad (2.2.14)$$

where  $A$  and  $C$  are constants and  $k_x = n\pi/L_x$  and  $k_y = m\pi/L_y$  where  $m$  and  $n$  are arbitrary integers. So the spatial variation of the drum's vibration must be proportional to

$$\sin\left(\frac{n\pi x}{L_x}\right) \sin\left(\frac{m\pi y}{L_y}\right). \quad (2.2.15)$$

Now since

$$\frac{\omega^2}{c^2} = k_x^2 + k_y^2$$

we have

$$\frac{\omega^2}{c^2} = \pi^2 \left( \frac{n^2}{L_x^2} + \frac{m^2}{L_y^2} \right) \quad (2.2.16)$$

As  $n$  and  $m$  vary over the integers,  $\omega$  defines a doubly-infinite set of resonant frequencies. The same argument we made before about initial conditions applies here. To be able to solve a general initial value problem we need to be able to represent the initial conditions in a Fourier series. This will be a 2-D Fourier series in  $x$  and  $y$  but that's not a big deal.

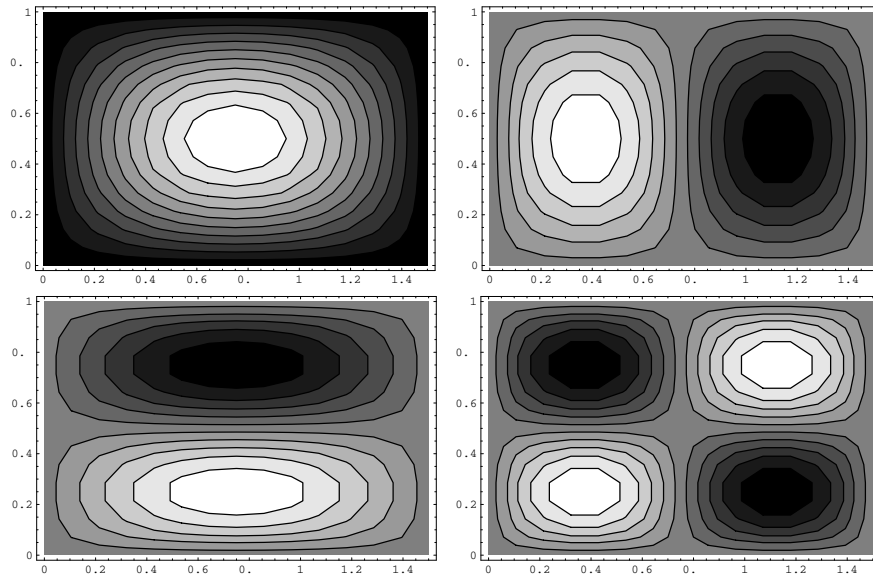


Figure 2.2: The first four modes of a rectangular drum. The aspect ratio of the drum is 1.5. The sides are clamped, so the displacement must be zero on the edge.

## 2.3 An Example

Here is a simple piece of *Mathematica* code that will draw the modes of a rectangular plate.

```

Lx = 1.5;
Ly = 1;
c = 1;
d[x_,y_,m_,n_] = Sin[m Pi x/Lx] Sin[n Pi y/Ly];
w[n_,m_] = c Sqrt[(m Pi /Lx)^2 + (n Pi/Ly)^2];

Do[
  Do[
    ContourPlot[d[x,y,m,n],{x,0,Lx},{y,0,Ly},
      AspectRatio->Ly/Lx];
    ,{m,2}];
  ,{n,2}];

```

The results of running this code are shown in Figure 2.2.

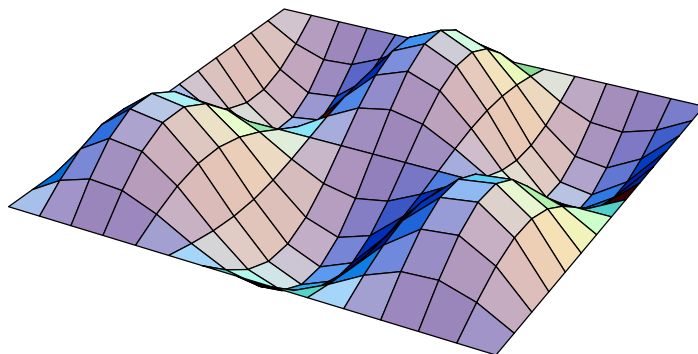


Figure 2.3: A perspective view of mode 3-2.

And in Figure 2.3 is a 3D perspective view of one of the modes. On the WWW page you'll find a *Mathematica* notebook that animates this.

## 2.4 Degeneracy

When we studied the Zeeman effect we saw that in the absence of a magnetic field, all three degrees of freedom oscillated with the same frequency. Applying a magnetic field splits this degenerate frequency into 3. The same thing happens with the drum. The expression we derived for the frequency of oscillation was

$$\omega_{n,m}^2 = c^2 \pi^2 \left( \frac{n^2}{L_x^2} + \frac{m^2}{L_y^2} \right).$$

Attaching the subscript to  $\omega$  is a good reminder that it depends on the mode. Now, clearly if  $L_x = L_y$ , then  $\omega_{i,j} = \omega_{j,i}$ . This is degeneracy. If  $L_x$  is just slightly different than  $L_y$ , then the frequencies are different. But even if the frequencies are the same, the modes  $n - m$  and  $m - n$  are clearly different. For example, in Figure 2.4 you will see plots of the modes  $n = 1 \quad m = 2$  and  $m = 1 \quad n = 2$  for a drum for which  $L_x = L_y = 1$ . The two modes have different vibrational patterns, but the same frequency.

Suppose we excited the drum at a frequency  $\omega_{12} = \omega_{21}$ ? What sort of pattern of nodal lines would we see? Like waves, modes will interfere constructively or destructively. This is a very interesting topic and we only touch upon it. But if the modes 12 and 21 were to constructively interfere, we would expect to see a vibrational pattern such as in Figure 2.5.

Finally we point out an interesting connection between number theory and normal modes. Let us rewrite our expression for the eigenfrequencies as

$$\omega_{n,m}^2 = \frac{c^2 \pi^2}{L_x^2} \left( n^2 + m^2 \frac{L_x^2}{L_y^2} \right).$$



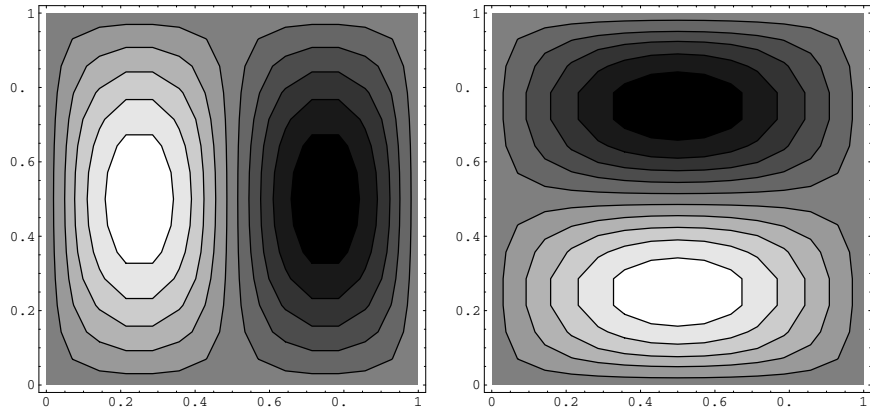


Figure 2.4: Modes 21 (left) and 12 (right) for a square drum.

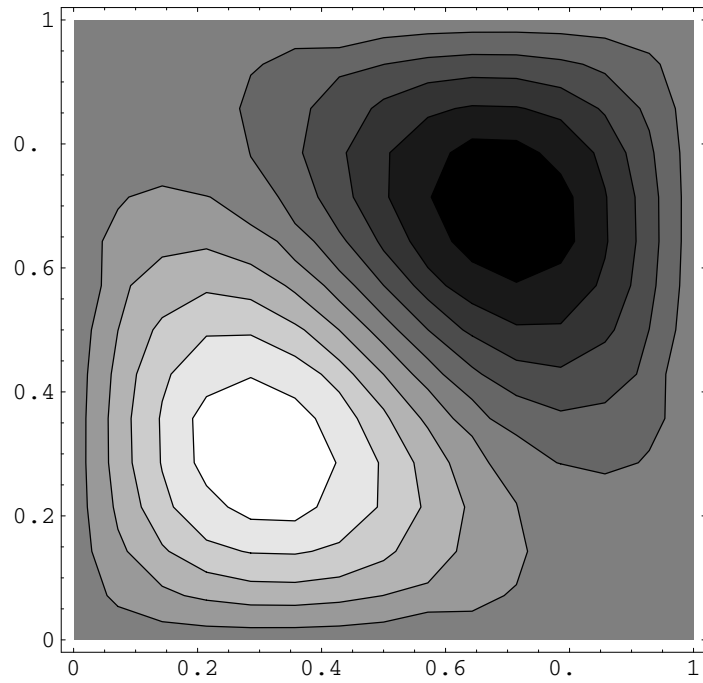


Figure 2.5: The sum of the two modes 12 and 21.

Let's suppose, just to keep life simple, that  $c$  is equal to  $L_x/\pi$ . And let's call the ratio  $\frac{L_x^2}{L_y^2} = \xi$ , so we have

$$\omega_{n,m}^2 = (n^2 + \xi m^2).$$

So the number-theoretic question I want to ask is: are there integers  $i, j$  and  $p, q$  such that the two frequencies  $\omega_{i,j}$  and  $\omega_{p,q}$  are equal? If they are equal then we have a degeneracy, if not, we don't. In other words, under what circumstances is it true that

$$p^2 + \xi q^2 = i^2 + \xi j^2?$$

Clearly this will be true if and only if

$$p^2 - i^2 = \xi(j^2 - q^2).$$

Assuming that  $j \neq q$  of course, this implies that

$$\frac{p^2 - i^2}{j^2 - q^2} = \xi.$$

Since all of the numbers  $p, q, i, j$  are integers, this equation can only be true if  $\xi$  is a rational number. Therefore we have proved that if the ratio of the lengths of the sides of the drum is irrational, then there is no degeneracy. The Greeks, who knew all about harmonics of music, described sides whose ratio was irrational as being “incommensurate”, a word that means not measurable. The Platonists had a theory that the universe was made of whole numbers. One would be a point. Two would be a line joining two points, three would be a triangle, and so on. They thought that everything could be built up out of these basic unit. It was a shock therefore to discover that the diagonal of a unit square was incommensurate: it could not be measured by any ruler made from the side of the square. No matter how finely you made the lines of the ruler, the diagonal would fall somewhere in between two lines. And not mid-way either, somewhere off to one side or the other.

## 2.5 Pictures of Modes

I'll conclude this chapter with some pictures of real modes. A “stadium” is a geometrical shape consisting of two circles on the end of a rectangle. The stadium shape has an important role in modern theories of chaos. Figure 2.6 shows two modes of such a shape visualized by dropping sand. The sand collects on node lines (i.e., places where the displacement is zero). These are called Chladni figures.

This particular stadium consists of an aluminum plate 194 mm long by 100 mm wide by about 3 mm thick. The plate is attached via a screw through a hole in the middle to a resonator. The whole system is driven by an HP function generator putting out a 10 volt

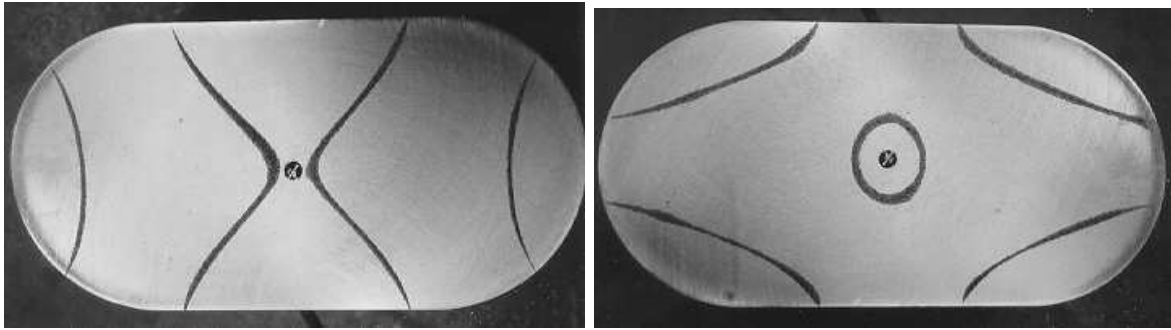


Figure 2.6: Chladni figures of two normal modes of a stadium plate. The mode on the left has a frequency of 1754 Hz and the one right 2116 Hz.

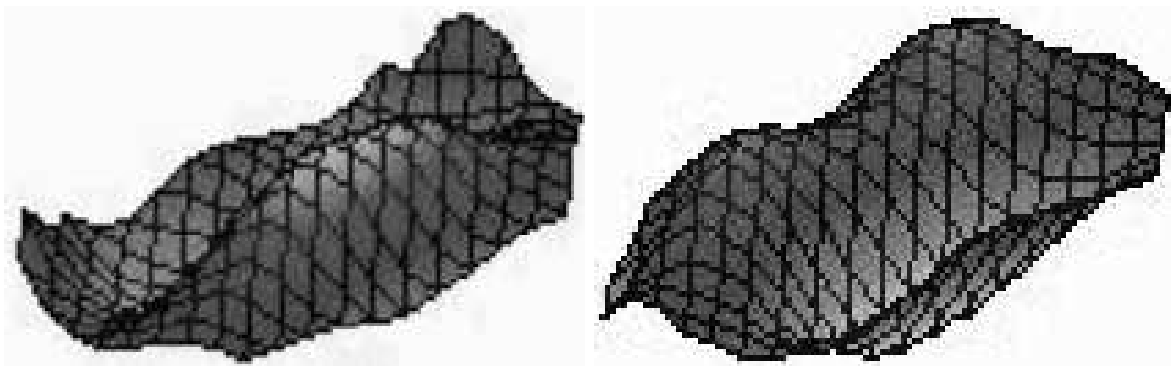


Figure 2.7: Displacement of the stadium plate at two different times when being driven in one of its normal modes. The measurement was performed with a laser-Doppler vibrometer.

RMS sine wave at the frequencies indicated in the caption. For more Chladni figures of the stadium plate, see my web page.

Figure 2.7 shows a different visualization of a mode of the stadium. These are two snapshots of the instantaneous displacement of the plane when being driven in one of its modes. The measurements were made with a laser-Doppler vibrometer.

## 2.6 Spherical and Cylindrical Harmonics

In this section we will apply separation of variables to Laplace's equation in spherical and cylindrical coordinates. Laplace's equation is important in its own right as the cornerstone of potential theory, but the wave equation also involves the Laplacian derivative, so the ideas discussed in this section will be used to build solutions of the wave equation in spherical and cylindrical coordinates too. The treatment given here is completely

standard and I have nothing new to say on the subject. The results are given here for convenience; but many excellent textbooks cover the same ground. A particular favorite of mine, for undergraduates, is *Classical Electromagnetic Radiation* by Heald and Marion [5].

Spherical coordinates are important when treating problems with spherical or nearly-spherical symmetry. To a first approximation the earth is spherical and so is the hydrogen atom, with lots of other examples in-between. Before we treat the wave equation, let's look at the simpler problem of Laplace's equation:

$$\nabla^2\psi(x, y, z) = 0. \quad (2.6.1)$$

In Cartesian coordinates this is:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \quad (2.6.2)$$

Laplace's equation is fundamental in geophysics since it describes the behavior of static electric and gravitational fields outside of the regions where this is charge or matter. For example, a point charge  $q$  at the origin creates an electric potential  $\psi(r) = \frac{q}{r}$ . As an exercise, carry out the differentiations of  $\frac{1}{r} = (x^2 + y^2 + z^2)^{-1/2}$  and show that  $\nabla^2\psi(x, y, z)$  is identically zero for  $r > 0$ , where the charge is located.



Joseph Louis Lagrange introduced the idea of potentials into his work on gravity. Lagrange is almost universally described as one of the great French mathematicians, but he was actually born in Turin (in what is now Italy) and baptized in the name of Giuseppe Lodovico Lagrangia. Lagrange, who worked for over 20 years in Berlin, made fundamental contributions in nearly all areas of mathematics and physics, in particular astronomy, the stability of the solar system, mechanics, dynamics, fluid mechanics, probability, and the foundations of the calculus as well as number theory. Lagrange died in Paris in April 1813.

This and other biographical material you will find in this book comes largely from the the St. Andrews University History of Mathematics WWW page:

<http://www-groups.dcs.st-andrews.ac.uk/~history/Mathematicians>.



Pierre-Simon Laplace really was French, having been born in Normandy in 1749. Laplace's mathematical talents were recognized early and he moved to Paris when he was 19 to further his studies. Laplace presented his first paper to the *Académie des Sciences* in Paris when he was 21 years old. He went on to make profound advances in differential equations and celestial mechanics. Laplace survived the reign of terror and was one of the first professors at the new *Ecole Normale* in Paris. Laplace propounded the nebular hypothesis for the origin of the solar system in his *Exposition du système du monde*. He also advanced the radical proposal that there could exist stars so massive that light could not escape them—we call these black holes now! And *Traité du Mécanique Céleste* is still print and widely read. Laplace also made fundamental contributions to mathematics, but I will mention only his book *Théorie Analytique des Probabilités*. He died on the third of March 1827 in Paris.

When solving boundary value problems for differential equations like Laplace's equation, it is extremely handy if the boundary on which you want to specify the boundary conditions can be represented by holding one of the coordinates constant. For instance, in Cartesian coordinates the surface of the unit cube can be represented by:

$$\begin{aligned} z &= \pm 1, & \text{for } -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1 \\ y &= \pm 1, & \text{for } -1 \leq z \leq 1 \text{ and } -1 \leq x \leq 1 \\ x &= \pm 1, & \text{for } -1 \leq z \leq 1 \text{ and } -1 \leq y \leq 1 \end{aligned}$$

On the other hand, if we tried to use Cartesian coordinates to solve a boundary value problem on a spherical domain, we couldn't represent this as a fixed value of any of the coordinates. Obviously this would be much simpler if we used spherical coordinates, since then we could specify boundary conditions on, for example, the surface  $r = \text{constant}$ . The disadvantage to using coordinate systems other than Cartesian is that the differential operators are more complicated. To derive an expression for the Laplacian in spherical coordinates we have to change variables according to:  $x = r \cos \phi \sin \theta$ ,  $y = r \sin \phi \sin \theta$ ,  $z = r \cos \theta$ . The angle  $\theta$  runs from 0 to  $\pi$ , while the angle  $\phi$  runs from 0 to  $2\pi$ .

Here is the result, the Laplacian in spherical coordinates:

$$\nabla^2 \psi(x, y, z) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \quad (2.6.3)$$

### Physical interpretation of the Laplacian

In 1-dimension, Laplace's equation says:  $\phi''(x) = 0$ . This equation can be integrated to give:  $\phi(x) = ax + b$ . So in 1-D any linear function (or a constant) satisfies Laplace's equation. The Laplacian operator itself measures (more or less) the curvature of a function of space. So since Laplace's equation says that this must be zero, it stands to reason the harmonic functions would be relatively smooth.

## 2.6.1 separation of variables

Look for solutions of the form:  $\psi(r, \theta, \phi) = R(r)P(\theta)Q(\phi)$ . So,

$$\begin{aligned} \nabla^2 \psi(r, \theta, \phi) &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \\ &= \frac{PQ}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{RQ}{r^2 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dP}{d\theta} \right) + \frac{RP}{r^2 \sin^2 \theta} \frac{d^2 Q}{d\phi^2} = 0. \end{aligned} \quad (2.6.4)$$

Dividing, as usual, by  $RPQ$  we have:

$$\frac{1}{Rr^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{Pr^2 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dP}{d\theta} \right) + \frac{1}{Qr^2 \sin^2 \theta} \frac{d^2 Q}{d\phi^2} = 0. \quad (2.6.5)$$

This looks more complicated than what we had with rectangular coordinates. In fact it looks like we're stuck since all three terms involve both  $r$  and  $\theta$ . However, multiplying by  $r^2 \sin^2 \theta$  makes the third term just  $\frac{1}{Q} \frac{d^2 Q}{d\phi^2}$ . So,

$$\frac{\sin^2 \theta}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{\sin \theta}{P} \frac{d}{d\theta} \left( \sin \theta \frac{dP}{d\theta} \right) = -\frac{1}{Q} \frac{d^2 Q}{d\phi^2}. \quad (2.6.6)$$

This we can make some progress with since we have a function of  $r$  and  $\theta$  on the left side and a function of  $\phi$  on the right; therefore both sides must be equal to a constant, which we'll call  $m^2$ . Thus

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \quad (2.6.7)$$

and so  $Q$  must be proportional to  $e^{im\phi}$ . In order that the solution be continuous, we must require that  $Q(\phi) = Q(\phi + 2\pi)$  so  $m$  must be an integer. Of course, it may happen that one is interested in solving a boundary on a subset of a sphere, in which case it may not be true that  $Q$  is continuous; in that case  $m$  need not be an integer.

Next, for the  $r$  and  $\theta$  part of Equation 2.6.6.

$$\frac{\sin^2 \theta}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = -\frac{\sin \theta}{P} \frac{d}{d\theta} \left( \sin \theta \frac{dP}{d\theta} \right) + m^2. \quad (2.6.8)$$

Again, to separate the  $r$  and  $\theta$  we divide by  $\sin^2 \theta$ :

$$\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = -\frac{1}{P \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dP}{d\theta} \right) + \frac{m^2}{\sin^2 \theta}. \quad (2.6.9)$$

Now we can introduce another separation constant, call it  $k^2$ . With this we get the radial equation:

$$\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = k^2. \quad (2.6.10)$$

This turns out to be easy to solve if we guess a trial solution of the form  $R = Ar^\alpha$ . Plugging this into the radial equation we get:

$$\alpha(\alpha + 1)Ar^\alpha - k^2 Ar^\alpha = 0 \quad (2.6.11)$$

which implies that  $k^2 = \alpha(\alpha + 1)$ . Now if we were to write the separation constant  $k^2$  as  $k^2 = \ell(\ell + 1)$ , then it would be easy to see that

$$\alpha(\alpha + 1) = \ell(\ell + 1) \quad (2.6.12)$$

is the same as

$$(\alpha - \ell)(\alpha + (\ell + 1)) = 0. \quad (2.6.13)$$

This equation is satisfied for  $\alpha = \ell$  and  $\alpha = -(\ell + 1)$ . In other words, the solution to

$$\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = \ell(\ell + 1) \quad (2.6.14)$$

is

$$R(r) = A_\ell r^\ell + B_\ell r^{-(\ell+1)}. \quad (2.6.15)$$

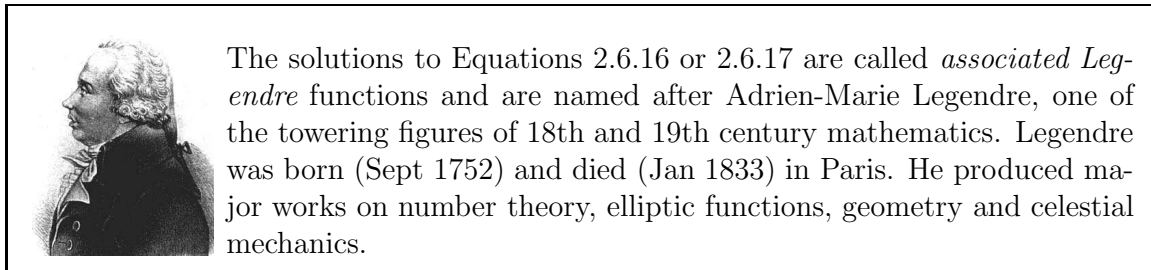
Lastly, we must solve the  $\theta$  equation for  $P$  (i.e., the right side of Equation 2.6.9 set equal to  $\ell(\ell + 1)$ ):

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dP}{d\theta} \right) + \left[ \ell(\ell + 1) - \frac{m^2}{\sin^2 \theta} \right] P = 0. \quad (2.6.16)$$

This is called Legendre's equation and is sometimes written in terms of the variable  $x = \cos \theta$  since then,  $\frac{1}{\sin \theta} \frac{d}{d\theta} = -\frac{d}{dx}$  which leads to

$$\frac{d}{dx} \left[ (1 - x^2) \frac{dP}{dx} \right] + \left[ \ell(\ell + 1) - \frac{m^2}{1 - x^2} \right] P. \quad (2.6.17)$$

Remember that the angle  $\theta$  runs from 0 to  $\pi$ , so the same interval in  $x$  corresponds to  $[-1, 1]$ .



The standard way to proceed with the solution of Legendre's equation is by power series. The solution  $P$  is expanded in a power series in  $x$  (or  $\cos \theta$ ) of the form:

$$P(x) = (1 - x^2)^{m/2} \sum_{n=0}^{\infty} a_n x^n \quad (2.6.18)$$

Since the solutions must be defined on the interval  $[-1, 1]$ , we do not include any negative powers of  $x$ . So, to find the coefficients  $a_n$ , we insert the power series into Equation 2.6.17 and derive a recursion relation. I will skip the details, which you can find in many applied mathematics books, but the essential idea is that the power series solution diverges at the end-points  $x = \pm 1$  unless  $\ell \geq |m|$ . And in this case the power series actually terminates and becomes a polynomial in  $x$ : the coefficients  $a_n$  are zero when  $n > \ell - |m|$ . This is why the solutions are called Legendre polynomials; they are written  $P_{\ell m}(x)$ , with  $\ell$  and  $m$  integers and  $\ell \geq |m|$ . Strictly speaking  $P_{\ell m}(x)$  are called associated Legendre polynomials. The term Legendre polynomial is reserved to the special case of  $m = 0$ .

### The case of axial symmetry: $m = 0$

The separation constant  $m$  appears in both the  $\phi$  (i.e.,  $Q$ ) and  $\theta$  (i.e.,  $P$ ) equations. However, if  $m = 0$  then  $Q(\phi)$  is just a constant. So, for problems which are symmetrical about the  $z$  axis (independent of  $\phi$ ) the  $\theta$  equation reduces to

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dP}{d\theta} \right) + \ell(\ell + 1)P = 0 \quad (2.6.19)$$

while the  $x$  equation reduces to

$$(1 - x^2) \frac{d^2 P}{dx^2} - 2x \frac{dP}{dx} + \ell(\ell + 1)P = 0. \quad (2.6.20)$$

The solution depends on only one index now,  $\ell$ , and is written  $P_{\ell}(x)$ . By examining the recursion relation for the coefficients of the power series one can derive the following two formulae for the Legendre polynomials:



$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell. \quad (2.6.21)$$

$$P_{\ell m}(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x). \quad (2.6.22)$$

Of course  $P_\ell^m$  reduces to  $P_\ell$  when  $m = 0$ . These expressions for the Legendre polynomials are referred to as *Rodrigues' formulae*.<sup>2</sup>

So the separable solutions to Laplace's equation involves multiplying the  $r$  solutions by the  $\theta$  solutions by the  $\phi$  solutions:

$$\psi(r, \theta, \phi) = \left\{ \begin{array}{l} r^\ell \\ r^{-(\ell+1)} \end{array} \right\} P_{\ell m}(\cos \theta) e^{im\phi} \quad (2.6.23)$$

which reduces in the axi-symmetric case to

$$\psi(r, \theta, \phi) = \left\{ \begin{array}{l} r^\ell \\ r^{-(\ell+1)} \end{array} \right\} P_\ell(\cos \theta). \quad (2.6.24)$$

This is the final result of separation of variables. You will have to take it on faith for now that **any** solution of Laplace's equation can be built up by superposition out of these basic separation of variables solutions. In other words, **any potential function** (solution to Laplace's equation) can be written as:

$$\psi(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (A_{\ell m} r^\ell + B_{\ell m} r^{-(\ell+1)}) Y_{\ell m}(\theta, \phi). \quad (2.6.25)$$

Shortly, when we solve boundary value problems for Laplace's equation, all the work will boil down to computing the  $A$  and  $B$  coefficients given the boundary values.

Here are the first few Legendre polynomials:

$$P_1(x) = 1 \quad (2.6.26)$$

$$P_2(x) = x \quad (2.6.27)$$

$$P_3(x) = \frac{1}{2}(3x^2 - 1) \quad (2.6.28)$$

It is standard to put the  $\theta$  and  $\phi$  dependence of the solutions to Laplace's equations together into a single set of functions called *spherical harmonics*.<sup>3</sup> The spherical harmonics

<sup>2</sup>Rodrigues was actually Benjamin Olinde, born in Bordeaux in 1794, the son of a wealthy Jewish banker. Olinde studied mathematics at the *Ecole Normale* in Paris, taking his doctor's degree in 1816 with a thesis containing the famous formulae for Legendre polynomials.

<sup>3</sup>A harmonic is any solution of Laplace's equation.

are defined as:

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} P_{\ell m}(\theta, \phi) e^{im\phi}. \quad (2.6.29)$$

The first few spherical harmonics are:

$$Y_{00}(\theta, \phi) = \sqrt{\frac{1}{4\pi}} \quad (2.6.30)$$

$$Y_{10}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta \quad (2.6.31)$$

$$Y_{1\pm 1}(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi} \quad (2.6.32)$$

$$Y_{20}(\theta, \phi) = \sqrt{\frac{5}{16\pi}} (2 \cos^2 \theta - \sin^2 \theta) \quad (2.6.33)$$

$$Y_{2\pm 1}(\theta, \phi) = \mp \sqrt{\frac{15}{8\pi}} \cos \theta \sin \theta e^{\pm i\phi} \quad (2.6.34)$$

$$Y_{2\pm 2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi} \quad (2.6.35)$$

## 2.6.2 Properties of Spherical Harmonics and Legendre Polynomials

The Legendre polynomials and the spherical harmonics satisfy the following “orthogonality” relations. We will see shortly that these properties are the analogs for functions of the usual orthogonality relations you already know for vectors.

$$\int_{-1}^{-1} P_{\ell'}(x) P_{\ell}(x) dx = \frac{2}{2\ell + 1} \delta_{\ell\ell'} \quad (2.6.36)$$

$$\int_{-1}^{-1} P_{\ell'm}(x) P_{\ell m}(x) dx = \frac{2}{2\ell + 1} \frac{(\ell + m)!}{(\ell - m)!} \delta_{\ell\ell'} \quad (2.6.37)$$

$$\int_{4\pi} Y_{\ell m}(\theta, \phi) \bar{Y}_{\ell' m'}(\theta, \phi) d\Omega = \int_0^{2\pi} \int_0^{\pi} Y_{\ell m}(\theta, \phi) \bar{Y}_{\ell' m'}(\theta, \phi) \sin \theta d\theta d\phi = \delta_{\ell\ell'} \delta_{mm'} \quad (2.6.38)$$

where the over-bar denotes complex conjugation and  $\Omega$  represents solid angle:  $d\Omega \equiv \sin\theta d\theta d\phi$ . Using  $4\pi$  as the limit of integration is symbolic of the fact that if you integrate  $d\Omega$  over the sphere ( $\theta$  going from 0 to  $\pi$  and  $\phi$  going from 0 to  $2\pi$ ) you get  $4\pi$ . Notice that the second relation is slightly different than the others; it says that for any given value of  $m$ , the polynomials  $P_{\ell m}$  and  $P_{\ell' m}$  are orthogonal.

There is also the following “parity” property:

$$Y_{\ell m}(\pi - \theta, \phi + \pi) = (-1)^\ell Y_{\ell m}(\theta, \phi). \quad (2.6.39)$$

### orthogonal function expansions

The functions  $P_\ell(x)$  have a very special property. They are *complete* in the set of functions on  $[-1, 1]$ . This means that *any* (reasonable) function defined on  $[-1, 1]$  can be represented as a superposition of the Legendre polynomials:

$$f(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x). \quad (2.6.40)$$

To compute the coefficients of this expansion we use the orthogonality relation exactly as you would with an ordinary vector. For example, suppose you want to know the  $x$ -component of a vector  $\mathbf{T}$ . All you have to do is take the inner product of  $\mathbf{T}$  with  $\hat{\mathbf{x}}$ . This is because

$$\mathbf{T} = T_x \hat{\mathbf{x}} + T_y \hat{\mathbf{y}} + T_z \hat{\mathbf{z}}$$

so

$$\hat{\mathbf{x}} \cdot \mathbf{T} = T_x \hat{\mathbf{x}} \cdot \hat{\mathbf{x}} + T_y \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} + T_z \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = T_x$$

since  $\hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0$  and  $\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = 1$ . When you take the inner product of two vectors you sum the product of their components. The analog of this for functions is to sum the product of the values of the function at each point in their domains. Since the variables are continuous, we use an integration instead of a summation. So the “dot” or inner product of two functions  $f(x)$  and  $g(x)$  defined on  $[-1, 1]$  is:

$$(f, g) = \int_{-1}^1 f(x)g(x)dx. \quad (2.6.41)$$

So, to find the expansion coefficients of a function  $f(x)$  we take the inner product of  $f$  with each of the Legendre “basis vectors”  $P_\ell(x)$ :

$$\int_{-1}^1 f(x)P_{\ell'}(x)dx = \sum_{\ell=0}^{\infty} A_\ell \int_{-1}^1 P_\ell(x)P_{\ell'}(x)dx = \sum_{\ell=0}^{\infty} A_\ell \frac{2}{2\ell+1} \delta_{\ell\ell'} = \frac{2A_{\ell'}}{2\ell'+1}. \quad (2.6.42)$$

So, the  $\ell$ -th coefficient of the expansion of a function  $f(x)$  is

$$A_\ell = \frac{2\ell+1}{2} \int_{-1}^1 f(x)P_\ell(x)dx \quad (2.6.43)$$

Similarly, we can expand any function defined on the surface of the unit sphere in terms of the  $Y_{\ell m}(\theta, \phi)$ :

$$\psi(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} Y_{\ell m}(\theta, \phi) \quad (2.6.44)$$

with expansion coefficients

$$A_{\ell m} = \int_{4\pi} \psi(\theta, \phi) \bar{Y}_{\ell m}(\theta, \phi) d\Omega. \quad (2.6.45)$$

For example, what is the expansion in spherical harmonics of 1? Only the  $\ell = 0, m = 0$  spherical harmonic is constant, so

$$1 = \sqrt{4\pi} Y_{00}.$$

In other words,  $A_{\ell m} = \sqrt{4\pi} \delta_{0,0}$ .

### What is a field?

The term “field” is used to refer to any function of space. This could be a scalar function or it could be a vector or even tensor function. Examples of scalar fields include: temperature, acoustic pressure and mass density. Examples of vector fields include the electric and magnetic fields, gravity, elastic displacement. Examples of tensor fields include the stress and strain inside continuous bodies.

## 2.7 Exercises

1. Apply separation of variables to Laplace’s equation in cylindrical coordinates:

$$\nabla^2 \psi(r, \theta, z) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} = 0.$$

**answer:** We make the, by now, standard assumption that we can write the solution in the form  $\psi(r, \theta, z) = R(r)Q(\theta)Z(z)$ . Plugging this into Laplace’s equation and dividing by  $RQZ$  we have:

$$\frac{1}{Rr} \frac{d}{dr} \left( r \frac{dR}{dr} \right) + \frac{1}{r^2 Q} \frac{d^2 Q}{d\theta^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0. \quad (2.7.1)$$

At this point we have a choice as to the order of the solution. We could first isolate the  $z$  equation or we could isolate the  $\theta$  equation. Also, in choosing the sign of the separation constant, we are in effect choosing whether we want an exponentially decaying solution or a sinusoidal one. I suggest isolating the  $\theta$  equation first since

we almost always want our solutions to be continuous in angle. That means we expect the fundamental  $\theta$  dependence to be sinusoidal, so we write

$$\frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right) + \frac{r^2}{Z} \frac{d^2 Z}{dz^2} = -\frac{1}{Q} \frac{d^2 Q}{d\theta^2} = m^2. \quad (2.7.2)$$

This gives us

$$\frac{d^2 Q}{d\theta^2} + m^2 Q = 0 \quad (2.7.3)$$

which has solutions proportional to  $e^{\pm im\theta}$ . Now if we had chosen the separation constant to be  $-m^2$ , then we would have gotten

$$\frac{d^2 Q}{d\theta^2} - m^2 Q = 0 \quad (2.7.4)$$

the solutions of which are proportional to  $e^{\pm m\theta}$ . Since we usually don't expect exponential decay with angle, we choose the plus sign for the separation constant. As we will see shortly, the choice is less clear cut for the other variables. In any case we now have for the  $r, \theta$  dependence:

$$\frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right) - m^2 = \frac{r^2}{Z} \frac{d^2 Z}{dz^2} \quad (2.7.5)$$

or

$$\frac{1}{Rr} \frac{d}{dr} \left( r \frac{dR}{dr} \right) - \frac{m^2}{r^2} = -\frac{1}{Z} \frac{d^2 Z}{dz^2}. \quad (2.7.6)$$

Once again we must decide on the sign of the separation constant. Looking at the  $z$  equation we could imagine either sinusoidal or exponential dependence. So let's do both cases. First let's look for exponential  $z$  dependence. That means we'll need a negative separation constant, say,  $-k^2$ :

$$\frac{1}{Rr} \frac{d}{dr} \left( r \frac{dR}{dr} \right) - \frac{m^2}{r^2} = -\frac{1}{Z} \frac{d^2 Z}{dz^2} = -k^2, \quad (2.7.7)$$

which implies

$$\frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right) + r^2 k^2 - m^2 = 0 \quad (2.7.8)$$

and

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0. \quad (2.7.9)$$

The  $Z$  solutions are now proportional to  $e^{\pm iz}$ .

Now for the  $R$  solutions. These satisfy

$$r \frac{d}{dr} \left( r \frac{dR}{dr} \right) + (r^2 k^2 - m^2) R = 0$$

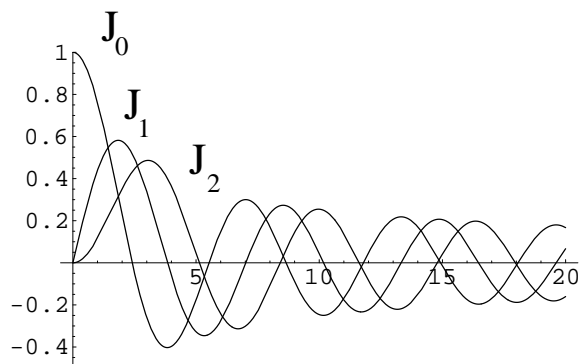


Figure 2.8: The first 3 cylindrical Bessel functions.

The constant  $k$  arose from  $z$ -separation, so if we set  $k = 0$  this corresponds to no  $z$ -dependence. This is useful, for example, in studying the potentials of long wires, where we can neglect the  $z$ -dependence. It is easy to show, similarly to what we did with the axi-symmetric spherical harmonics, that in the case  $k = 0$ , the radial solutions are of the form  $r^m$  and  $r^{-m}$  (for  $m > 0$ ). To see this just make a trial solution of the form  $R(r) = Ar^\alpha$ , then show that this satisfies the radial equation if and only if  $\alpha^2 = m^2$ .

The radial equation above is almost in the standard form of Bessel's equation.



Friedrich Wilhelm Bessel (born 22 July 1784 in Minden, Westphalia, died 17 March 1846 in Königsberg, Prussia) was a leading figure in 19th century astronomy. Bessel made fundamental advances in the calculation of planetary orbits and is also well-known for his work as a teacher and educational reformer.

To get it in the standard form we make the substitution:  $u = kr$ , then

$$\frac{1}{u} \frac{d}{du} \left( u \frac{dR}{du} \right) + \left( 1 - \frac{m^2}{u^2} \right) R = 0 \quad (2.7.10)$$

or after multiplication by  $u^2$ :

$$u^2 \frac{d^2 R}{du^2} + u \frac{dR}{du} + (u^2 - m^2) R = 0 \quad (2.7.11)$$

Solutions of this last equation are called cylindrical Bessel functions and are denoted by  $J_m(u)$ .

The power series solution to Bessel's equation can be found in many textbooks on differential equations and electricity and magnetism such as [5]. Here I will just

quote the result:

$$J_m(u) = \frac{u^m}{2^m m!} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{2^{2\ell} \ell! (m+1)(m+2)\cdots(m+\ell)} u^{2\ell} \quad (2.7.12)$$

This is hard to deal with analytically. However for small and large values of  $u = kr$ , there are nice analytic approximations:

$$J_m(kr) \approx \frac{1}{(m+1)!} \left(\frac{kr}{2}\right)^m \quad kr \ll 1 \quad (2.7.13)$$

and

$$J_m(kr) \approx \sqrt{\frac{2}{\pi kr}} \cos\left(kr - \frac{m\pi}{2} - \frac{\pi}{4}\right) \quad kr \gg 1 \quad (2.7.14)$$

### sinusoidal z-dependence and modified Bessel functions

On the other hand, if instead of choosing the separation constant to be  $-k^2$  we had chosen  $k^2$ , then

$$\frac{1}{Rr} \frac{d}{dr} \left( r \frac{dR}{dr} \right) - \frac{m^2}{r^2} = -\frac{1}{Z} \frac{d^2 Z}{dz^2} = k^2, \quad (2.7.15)$$

and we would have gotten  $Z$  solutions proportional to  $e^{\pm ikz}$  and the radial equation would have been:

$$r \frac{d}{dr} \left( r \frac{dR}{dr} \right) - (k^2 r^2 + m^2) R = 0 \quad (2.7.16)$$

or equivalently

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} - (k^2 r^2 + m^2) R = 0. \quad (2.7.17)$$

The solutions of this equation are called modified Bessel functions (of the first kind).

2. Expand  $f(x) = e^{-|x|}$  on  $[-1, 1]$  in terms of  $\ell = 0, 1, 2$  Legendre polynomials.

**answer:**  $A_0 = 1 - \frac{1}{e}$ .  $A_1 = 0$ .  $A_2 = \frac{25}{2} - \frac{35}{e}$ .

3. A grounded conducting sphere of radius  $a$  is placed in a plane parallel electric field  $\mathbf{E} = E_0 \hat{\mathbf{z}}$ . What is the electric field outside the conductor?

**answer:** First we will compute the potential  $V(r, \theta, \phi)$  then we will take the gradient of this to get the electric field. Since the electric field is axisymmetric about the  $z$ -axis, in fact the potential does not depend on  $\phi$ . So we can be sure that we can write the unknown potential as:

$$V(r, \theta) = \sum_{\ell=0}^{\infty} (A_\ell r^\ell + B_\ell r^{-(\ell+1)}) P_\ell(\cos \theta).$$

We have two boundary conditions that we can apply. First the potential on the surface is zero, so

$$V(r = a, \theta) = \sum_{\ell=0}^{\infty} (A_{\ell} a^{\ell} + B_{\ell} a^{-(\ell+1)}) P_{\ell}(\cos \theta) = 0.$$

The second boundary condition is that as  $r \rightarrow \infty$ , the potential must approach that of the original, unperturbed E-field:

$$\lim_{r \rightarrow \infty} V(r, \theta) = -E_0 z$$

where  $\mathbf{E} = -\nabla(-E_0 z)$ . If we apply the orthogonality condition to the  $r = a$  boundary condition, we can see that

$$\begin{aligned} 0 &= \sum_{\ell=0}^{\infty} (A_{\ell} a^{\ell} + B_{\ell} a^{-(\ell+1)}) \int_{-1}^1 P_{\ell}(x) P_{\ell'}(x) dx \\ &= \sum_{\ell=0}^{\infty} (A_{\ell} a^{\ell} + B_{\ell} a^{-(\ell+1)}) \frac{2}{2\ell + 1} \delta_{\ell\ell'} \\ &= A_{\ell} a^{\ell} + B_{\ell} a^{-(\ell+1)}. \end{aligned} \tag{2.7.18}$$

So we end up with the constraint that:  $B_{\ell} = -a^{2\ell+1} A_{\ell}$ .

Next we apply the large- $r$  condition. In the limit of large  $r$ , our boundary condition only constrains terms involving positive power of  $r$ , since the negative powers of  $r$  go to zero. So we must have

$$\lim_{r \rightarrow \infty} -E_0 r \cos \theta \equiv \lim_{r \rightarrow \infty} -E_0 r P_1(\cos \theta) = \sum_{\ell=0}^{\infty} A_{\ell} r^{\ell} P_{\ell}(\cos \theta).$$

It is clear from this that we can satisfy the boundary condition at infinity only if all the of  $A$  coefficients are zero except the  $\ell = 1$  term. So  $A_{\ell} = 0$  for all  $\ell$  except 1, and  $A_1 = -E_0$ . We combine this with the constraint we found on the  $A$  and  $B$  coefficients above to get:  $B_1 = -A_1 a^3 = E_0 a^3$ . With the result that the potential everywhere outside the sphere is:

$$V(r, \theta) = -E_0 r \cos \theta + E_0 a^3 \frac{\cos \theta}{r^2} = -E_0 \left( 1 - \left( \frac{a}{r} \right)^3 \right) r \cos \theta.$$

From this it follows by taking the gradient in spherical coordinates that:

$$E_r = E_0 \left( 1 + 2 \left( \frac{a}{r} \right)^3 \right) \cos \theta$$

and

$$E_{\theta} = -E_0 \left( 1 - \left( \frac{a}{r} \right)^3 \right) \sin \theta.$$



4. A grounded, spherical conductor of radius  $a$  is placed in an electric field such that far away from the sphere the potential is  $V(r, \theta, \phi) = r^2 \sin 2\theta \cos \phi$ . Find the potential everywhere outside the sphere.

**answer:** We can write any solution of Laplace's equation as:

$$\psi(r, \theta, \phi) = \sum_{\ell, m} (A_{\ell m} r^\ell + B_{\ell m} r^{-(\ell+1)}) Y_{\ell m}(\theta, \phi). \quad (2.7.19)$$

In this case we are told that far away from the conductor the potential is:  $r^2 \sin 2\theta \cos \phi$ . OK, for large  $r$  we can only say something about the  $A$  coefficients since the terms involving the  $B$  coefficients decay at least as fast as  $1/r$ . Of the  $A$  coefficients it is clear that since the field must be proportional to  $r^2$  for large  $r$ , only the  $\ell = 2$  terms can be nonzero. So straight away we can see that for large  $r$  the field must be of the form

$$\psi(r \rightarrow \infty, \theta, \phi) = r^2 (A_{22} Y_{22} + A_{21} Y_{21} + A_{20} Y_{20} + A_{2-1} Y_{2-1} + A_{2-1} Y_{2-2}). \quad (2.7.20)$$

If you look at the  $\ell = 2$  spherical harmonics you will see that only the  $m = 1$  terms are needed:

$$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$$

and

$$Y_{2-1} = \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{-i\phi}$$

so

$$\sqrt{\frac{8\pi}{15}} (Y_{2-1} - Y_{21}) = 2 \sin \theta \cos \theta \cos \phi.$$

In fact, since  $\sin 2\theta = 2 \sin \theta \cos \theta$ , it follows that:

$$r^2 \sin 2\theta \cos \phi = \sqrt{\frac{8\pi}{15}} r^2 (-Y_{21} + Y_{2-1}). \quad (2.7.21)$$

Therefore  $Y_{21} = -A_{2-1} = -\sqrt{\frac{8\pi}{15}}$ .

Now just as we did in the previous problem, we can apply the boundary condition that  $\psi(r = a, \theta, \phi) = 0$  to give a constraint on the  $A$  and  $B$  coefficients:

$$B_{\ell m} = -a^{2\ell+1} A_{\ell m}. \quad (2.7.22)$$

Hence only  $B_{21} = a^5 \sqrt{\frac{8\pi}{15}}$  and  $B_{2-1} = -a^5 \sqrt{\frac{8\pi}{15}}$  are nonzero. So now we have all four nonzero coefficients in the spherical harmonic expansion of  $\psi$ :

$$\begin{aligned}\psi(r, \theta, \phi) &= \sqrt{\frac{8\pi}{15}} \left[ \left( -r^2 + \frac{a^5}{r^3} \right) Y_{21} + \left( r^2 - \frac{a^5}{r^3} \right) Y_{2-1} \right] \\ &= \sqrt{\frac{8\pi}{15}} \left( r^2 - \frac{a^5}{r^3} \right) (-Y_{21} + Y_{2-1})\end{aligned}\quad (2.7.23)$$

$$= \left( 1 - \left( \frac{a}{r} \right)^5 \right) r^2 \sin 2\theta \cos \phi. \quad (2.7.24)$$

Notice that this agrees with the boundary condition when  $r = a$ . Always check your results against what you know.

5. Suppose the potential is constant on a sphere of radius  $a$ :  $\psi(r = a, \theta, \phi) = V_0$ . Use the spherical harmonic expansion of Laplace's equation to find the potential everywhere on the exterior of the sphere.

**answer:** On the surface of the sphere, the potential is a constant. Only the  $\ell = m = 0$  spherical harmonic is constant so

$$\psi(r = a, \theta, \phi) = V_0 = V_0 \sqrt{4\pi} Y_{00}. \quad (2.7.25)$$

This means that only the  $\ell = m = 0$  term in the expansion of the field is present. This tells us immediately that

$$\psi(r, \theta, \phi) = (A_{00} + B_{00} r^{-1}) Y_{00}. \quad (2.7.26)$$

Usually we don't care about constant potentials since they don't contribute to the electric or gravitational fields (the gradient of a constant is zero). So we can always shift the potential by a constant amount without changing physics; this means that we can ignore the  $A_{00}$  term. At  $r = a$  we have:

$$\psi(r = a, \theta, \phi) = B_{00} a^{-1} Y_{00} = V_0 \sqrt{4\pi} Y_{00}, \quad (2.7.27)$$

so  $B_{00} = V_0 a^{-1} \sqrt{4\pi}$  and the complete potential outside the sphere is

$$\psi(r, \theta, \phi) = (V_0 a \sqrt{4\pi}) r^{-1} Y_{00} = \frac{a V_0}{r}. \quad (2.7.28)$$

6. Consider the gravitational potential on the Earth's surface. The Earth is not exactly a sphere. A better approximation is:

$$\psi(r = R, \theta, \phi) = V_0 (1 - J_2 P_2(\cos \theta))$$

where  $J_2$  and  $V_0$  are constants. This is a bit of a trick actually since we're still assuming the surface is a sphere. What is the potential for  $r > R$ ?

**answer:** On the surface  $r = R$  the potential depends only on  $\theta$ , so

$$\psi(r = R, \theta) = V_0 (1 - J_2 P_2(\cos \theta)). \quad (2.7.29)$$

Since this problem is axi-symmetric (no  $\phi$ -dependence), the complete solution of Laplace's equation is

$$\psi(r, \theta) = \sum_{\ell=0}^{\infty} (A_{\ell} r^{\ell} + B_{\ell} r^{-(\ell+1)}) P_{\ell}(\cos \theta). \quad (2.7.30)$$

Applying the boundary condition we have

$$V_0 (1 - J_2 P_2(\cos \theta)) = \sum_{\ell=0}^{\infty} (A_{\ell} R^{\ell} + B_{\ell} R^{-(\ell+1)}) P_{\ell}(\cos \theta). \quad (2.7.31)$$

Using the orthogonality of the Legendre polynomials this equation implies two constraints on the  $A$  and  $B$  coefficients:

$$V_0 = A_0 + B_0 R^{-1}$$

and

$$-J_2 V_0 = (A_2 R^2 + B_2 R^{-3}).$$

The gravitational potential of the Earth cannot grow as you go farther away from the surface, so the  $A_2$  term must be zero. And as before we can set any constant potential term to zero. So we're left with:  $B_0 = R V_0$  and  $B_2 = -J_2 V_0 R^3$ . Which gives for the final solution:

$$\psi(r, \theta, \phi) = \frac{V_0 R}{r} \left[ 1 - J_2 \left( \frac{R}{r} \right)^2 P_2(\cos \theta) \right]. \quad (2.7.32)$$

The term  $J_2$  corresponds to the flattening of the Earth. If this term is zero, we're left with the usual  $1/r$  potential which equals  $V_0$  on the surface. In any case, the effects of the  $J_2$  term decay like  $1/r^2$  as you recede from the Earth's surface.

7. Consider two concentric, conducting spheres of radius  $r_0$  and  $r_1$  respectively. The inner sphere is grounded while the outer sphere is held at a fixed potential  $V_0$ . Find the potential between the spheres by directly integrating Laplace's equation in spherical coordinates. Hint: this problem is spherically symmetric.

**answer:** The spherical symmetry implies that the solution depends only on  $r$ . Therefore Laplace's equation reduces to

$$\nabla^2 \phi(r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) = 0.$$

This implies that  $r^2 \frac{\partial \phi}{\partial r}$  is a constant; call it  $c$ . Integrating once more we have

$$\phi(r) = d - \frac{c}{r}$$

where  $r$  is the second integration constant. Applying the two boundary conditions we can see that  $c = \frac{V_0(r_1 r_0)}{r_1 - r_0}$  and  $d = \frac{c}{r}$ . The final result is that

$$\phi(r) = V_0 \frac{r_1}{r} \frac{r - r_0}{r_1 - r_0} = V_0 \left[ \frac{1 - \frac{r_0}{r}}{1 - \frac{r_0}{r_1}} \right].$$

8. Two functions  $f(x)$  and  $g(x)$  defined on an interval  $[-1, 1]$  are said to be orthogonal on that interval if  $(f, g) \equiv \int_{-1}^1 f(x)g(x)dx = 0$ . Similarly we can define the squared “length” of a function on the interval by:  $(f, f) = \int_{-1}^1 f^2(x)dx$ . Here are two functions (polynomials of zeroth and first order)  $Q_0(x) = \sqrt{\frac{1}{2}}$  and  $Q_1(x) = \sqrt{\frac{3}{2}}x$ , that are orthogonal and have unit length. Compute the unique quadratic function  $Q_2(x)$  by using the three conditions:

$$\begin{aligned} (Q_0, Q_2) &= 0 \\ (Q_1, Q_2) &= 0 \\ (Q_2, Q_2) &= 1 \end{aligned}$$

**answer:** Since  $Q_2$  is a quadratic, it can be written  $Q_2(x) = ax^2 + bx + c$ . The conditions  $(Q_0, Q_2) = 0$  and  $(Q_1, Q_2) = 0$  force  $b = 0$  and  $c = -1/3a$ . The normalization condition  $(Q_2, Q_2)$  gives  $c = \frac{3}{2}\sqrt{\frac{5}{2}}$ . So,

$$Q_2(x) = \frac{3}{2}\sqrt{\frac{5}{2}} \left( x^2 - \frac{1}{3} \right).$$

9. • Give the spherical harmonic expansion of  $\sin \theta \cos \phi$ .  
• If this is the potential on a conducting sphere of radius 1, what is the potential for  $r > 1$ ?

**answer:** In the absence of any other fields, for solutions on the exterior of bodies we want potentials that decay with  $r$ . So the general solution must be of the form

$$\psi(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} r^{-(\ell+1)} Y_{\ell m}(\theta, \phi)$$

The boundary condition is:

$$\psi(r = 1, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} Y_{\ell m}(\theta, \phi) = \sin \theta \cos \phi.$$

But the right-hand side is

$$\sin \theta \cos \phi = \frac{1}{2} \sqrt{\frac{8\pi}{3}} (Y_{11} - Y_{1-1}).$$

So clearly only the  $\ell = 1$   $A$ -coefficients are nonzero:  $A_{11} = -A_{1-1} = \frac{1}{2}\sqrt{\frac{8\pi}{3}}$ . Hence, the potential outside the sphere is

$$\psi(r, \theta, \phi) = \frac{1}{2}\sqrt{\frac{8\pi}{3}}r^{-2}(Y_{11} - Y_{1-1}) = \frac{\sin\theta \cos\phi}{r^2}.$$

10. The 2D Laplace's equation in Cartesian coordinates is:

$$\nabla^2\psi(x, y) = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0.$$

Apply separation of variables to this equation to get two ordinary differential equations. Solve these equations. Explain how the choice of sign of the separation constant influences the solutions.

**answer:** By the now standard argument, we look for solutions of the form  $X(x)Y(y)$ , in which case Laplace's equation reduces to:

$$\frac{X''}{X} = -\frac{Y''}{Y}.$$

So we can choose the separation constant to be either  $k^2$  or  $-k^2$ . If we choose the plus sign, then the solution will be oscillatory in the  $y$  direction and exponential in the  $x$  direction. If we choose the negative sign, the solution will be oscillatory in the  $x$  direction and exponential in the  $y$  direction. E.g., with the positive sign we get

$$X'' - k^2X = 0$$

and

$$Y'' + k^2Y = -0.$$

So the basic solutions are of the form

$$\psi(x, y) = \sum_k e^{ikx} e^{-ky}$$

or

$$\psi(x, y) = \sum_k e^{-kx} e^{iky}$$

## 2.8 More on vectors

In the next chapter we will study vectors systematically, but you already know quite a lot about them. You were taught that a vector is something that has both a magnitude (length) and direction. Examples include gravity and the electric field. You also know

that any vector can be resolved into components. For example a vector  $\mathbf{T}$  in three dimensions can be resolved as

$$\mathbf{T} = T_x \hat{\mathbf{x}} + T_y \hat{\mathbf{y}} + T_z \hat{\mathbf{z}} \quad (2.8.1)$$

We will refer to the set of **all** three dimensional vectors like this as  $\mathbf{R}^3$  since it's really the real line  $\mathbf{R}^1$  in each of the three dimensions. The  $x - y$  plane is  $\mathbf{R}^2$ .

To find the components of the vector in the various directions we “dot” the basis vectors (i.e.,  $\hat{\mathbf{x}}$ ,  $\hat{\mathbf{y}}$  and  $\hat{\mathbf{z}}$ ) into  $\mathbf{T}$ . For example

$$\hat{\mathbf{x}} \cdot \mathbf{T} = T_x \hat{\mathbf{x}} \cdot \hat{\mathbf{x}} + T_y \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} + T_z \hat{\mathbf{x}} \cdot \hat{\mathbf{z}}. \quad (2.8.2)$$

But since the basis vectors are mutually perpendicular (orthogonal)  $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = 0$  and  $\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = 1$ . So  $T_x = \hat{\mathbf{x}} \cdot \mathbf{T}$  and similarly for the other components.

You can see that what really counts are the components of the vector in the mutually orthogonal directions. It doesn't really matter what we call these directions so we could also write

$$\mathbf{T} = T_x \hat{\mathbf{x}} + T_y \hat{\mathbf{y}} + T_z \hat{\mathbf{z}} = T_1 \hat{\mathbf{e}}_1 + T_2 \hat{\mathbf{e}}_2 + T_3 \hat{\mathbf{e}}_3 = \sum_{i=1}^3 T_i \hat{\mathbf{e}}_i \quad (2.8.3)$$

The advantage of labeling the directions by numbers is that it frees us from the constraints of ordinary three-dimensional geometry. Consider a time-series. Suppose I record the noon temperature for three days. Here are the data: (15.3, 8.5, 11.0). I can pretend that these are three components of a vector in  $\mathbf{R}^3$ . The fact is there is no physical “daily temperature vector” but I can treat these three numbers as if they were components of a three-dimensional vector. Or they could be the first three samples of a seismic trace. And if three, why not four? Or five? It turns out that a set of numbers such as (15.3, 8.5, 11.0, 12.1, 14.3) has exactly the same sort geometrical properties in  $\mathbf{R}^5$  as (15.3, 8.5, 11.0) does in  $\mathbf{R}^3$ , it's just that I can't make a plot of this vector on a piece of paper. I could extend this approach to quite long vectors, such as all the samples in a typical seismic trace, which might number in the thousands. Suppose that

$$\mathbf{A} = (a_1, a_2, a_3 \dots a_{1000})$$

is a one thousand sample seismic trace and

$$\mathbf{B} = (b_1, b_2, b_3 \dots b_{1000})$$

is another. Anything you can do with a three-dimensional vector you can do with a thousand-dimensional vector such as  $\mathbf{A}$  and  $\mathbf{B}$ , except plot them. We can add two seismic traces component-wise just as you would add two force vectors:

$$\mathbf{A} + \mathbf{B} = \sum_{i=1}^{1000} A_i + B_i.$$

We can take the length of a seismic trace:

$$\|\mathbf{A}\|^2 = \sum_{i=1}^{1000} A_i^2,$$

which is just the Pythagorean theorem in a 1000-dimensional space.

We can even compute the “angle” between two traces: since  $\mathbf{A} \cdot \mathbf{B} = \|\mathbf{A}\| \|\mathbf{B}\| \cos \theta$  works for ordinary vectors, there is no reason we cannot extend the idea to these abstract vectors. Then, the angle between the two traces is *naturally*:

$$\begin{aligned} \cos \theta &= \frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{A}\| \|\mathbf{B}\|} \\ &= \frac{\sum_{i=1}^{1000} A_i B_i}{\sum_{i=1}^{1000} A_i^2 \sum_{i=1}^{1000} B_i^2} \end{aligned}$$

Don’t be put off by this apparently abstract notion. The point is we can manipulate things like seismic traces as vectors and gain tremendous advantage from our prior geometrical understanding in  $\mathbf{R}^3$ , even though we’re not in  $\mathbf{R}^3$  any more. And for the final stroke, you should not be surprised if I tell you that we need not limit ourselves to vectors of finite length. Consider for example a power series:

$$f(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Think of the powers of  $x$  here as being like our basis vectors  $\hat{e}_i$  in  $\mathbf{R}^3$ . Then the coefficients  $a_i$  are just like the coefficients  $A_i$  of our length-1000 time series above; there just happens to be an infinite number of them! Well, OK, here we **do** run into a minor spot of difficulty. For a finite length series  $\sum_{i=0}^N a_i x^i$  we don’t have to worry about convergence, but for infinite series like  $\sum_{i=0}^{\infty} a_i x^i$ , we do. Apart from that, we can **still** use our geometrical intuition even in spaces of infinite dimension.

Look again at the dot or inner product of two finite length vectors:

$$\mathbf{A} \cdot \mathbf{B} = \sum_{i=1}^{1000} A_i B_i$$

exactly as in  $\mathbf{R}^3$ . We can certainly use the same formula for the dot product of two infinite dimensional vectors:

$$\mathbf{A} \cdot \mathbf{B} = \sum_{i=1}^{\infty} A_i B_i$$

provided the sum converges. The dot product for functions is just like this except that we can’t use a summation, we must use an integration:

$$f(x) \cdot g(x) = \int f(x)g(x)dx$$

where the integration is over whatever interval the functions are defined. It would be unconventional to use the “dot” for the inner product of functions, although we could. The standard notation for the dot product of functions is  $(f, g)$ , thus

$$(f, g) \equiv \int f(x)g(x)dx.$$

So, when we said a few pages ago that

$$\int_{-1}^1 P_{\ell'}(x)P_{\ell}(x)dx = \frac{2}{2\ell + 1}\delta_{\ell\ell'}$$

this means that the dot product of any  $P_{\ell}$  with any other is zero. So,  $(P_0, P_1) = 0$  and  $(P_1, P_{23}) = 0$ . It really does make sense to visualize  $P_0$ ,  $P_1$  and so on, as orthogonal vectors in some infinite dimensional space. There is nothing super-complicated about this idea; it really is quite natural when you get used to it. And not only does it save you a lot of time in the long run, it also allows you to apply your geometrical insight to solve problems that would be very difficult otherwise to solve.

One last thing. If I say that

$$a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}}$$

it will come as no surprise to you that this implies that the coefficients of each basis vector must be equal  $a_x = b_x$ ,  $a_y = b_y$ , and  $a_z = b_z$ . This is easy to prove just by dotting the above equation with each basis vector in succession. But now suppose I say that

$$a_0 + a_1x = b_0 + b_1x?$$

Does this imply that  $a_0 = b_0$  and  $a_1 = b_1$ ? Yes and for exactly the same reason. The basis vectors are  $1 = P_0$  and  $x = P_1$  are mutually orthogonal (at least on  $[-1, 1]$ )

$$\int_{-1}^1 1 \cdot x dx = 0.$$

Another way to see this is to re-arrange the equation  $a_0 + a_1x = b_0 + b_1x$  as:

$$(a_0 - b_0) = (b_1 - a_1)x.$$

But  $x$  is completely arbitrary here, so the only way this equation can possibly be true for **all**  $x$  is if it really says  $0 = 0x$ , or  $a_0 = b_0$ , and  $a_1 = b_1$ .

This orthogonality of the basis vectors is why we could say in the HW problem on rotational ellipticity of the Earth that an equation such as

$$V_0(1 - J_2P_2(\cos\theta)) = \sum_{\ell=0}^{\infty} (A_{\ell}R^{\ell} + B_{\ell}R^{-(\ell+1)}) P_{\ell}(\cos\theta). \quad (2.8.4)$$



forces us to conclude that

$$V_0 = A_0 + B_0 R^{-1}$$

and

$$-J_2 V_0 = (A_2 R^2 + B_2 R^{-3}).$$

The first equation comes from equating terms proportional  $P_0$  (which equals 1) on both sides of equation 2.8.4 and the second comes from equating terms proportional to  $P_2$ . All the other  $P_\ell$  terms are zero on the left side of equation 2.8.4 so they must be zero on the right.