Waves and Blackbody radiation

Simple model of a laser

What physics do we need to understand for lasers?

Scalar wave equation: 1D and 3D

3D waves

Energy in EM waves

A simple linear resonator

The 3D resonator and blackbody radiation

Reading

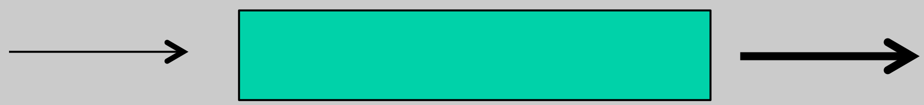
for today:

Svelto, Principles of Lasers, Ch1, 2.1, 2.2

for Wednesday: Svelto 2.3

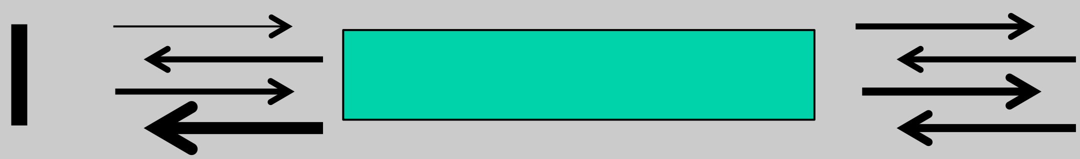
A simple model of a laser

- Stimulated emission leads to gain:



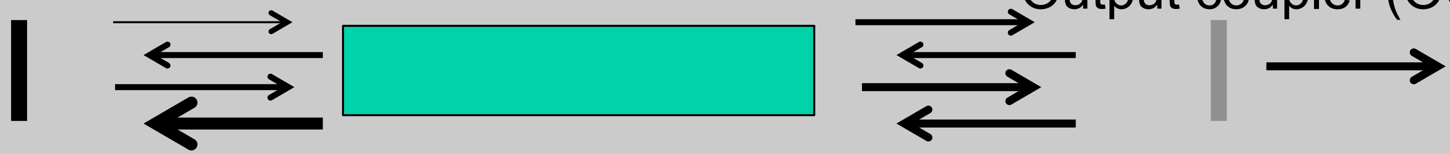
$$I_0 \propto E_0^2 \qquad E(z) = E_0 e^{+gz} \qquad I_1 \propto E_1^2 = I_0 e^{2gL}$$

- Add a resonator to give feedback:



$$I_n = I_0 \left(e^{2gL}\right)^n$$

- Leak some out for the output beam:



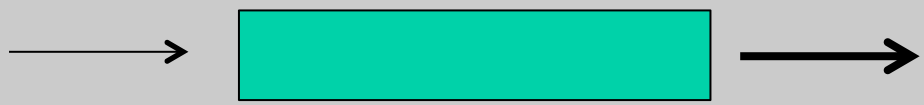
$$I_n = I_0 \left(e^{2gL}(1-T)\right)^n \equiv I_0 \left(e^{2gL} e^{-\gamma}\right)^n$$

Threshold: gain = loss

$$2gL - \gamma \qquad I_n = I_0$$

Overview of physics in lasers

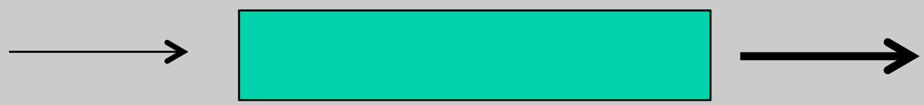
- Stimulated emission leads to gain:



$$I_0 \propto E_0^2 \qquad E(z) = E_0 e^{+gz} \qquad I_1 \propto E_1^2 = I_0 e^{2gL}$$

Overview of physics in lasers: light-matter interactions

- Stimulated emission leads to gain:



$$I_0 \propto E_0^2 \qquad E(z) = E_0 e^{+gz} \qquad I_1 \propto E_1^2 = I_0 e^{2gL}$$

  - How does stimulated emission work?
  - What conditions are necessary to get gain instead of absorption?
  - How do we get energy into the system? (pumping)
  - How do the properties of the atom (or other) affect the gain: spectrum, dynamics
  - What are different systems for getting gain?
    - Atoms, molecules, semiconductors, free-electrons…
  - What are the competing processes?

Overview of physics in lasers

- Add a resonator to give feedback:

$$I_n = I_0\left(e^{2gL}\right)^n$$

Overview of physics in lasers: resonators and beams

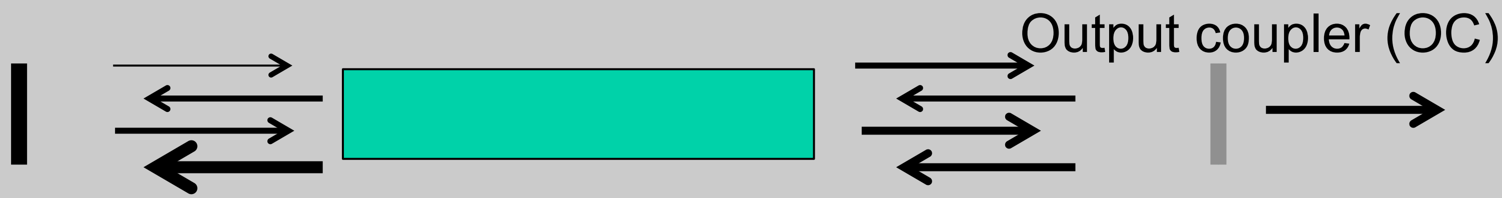
- Add a resonator to give feedback:

$$I_n = I_0\left(e^{2gL}\right)^n$$

  - How do we design the optics of the resonator to avoid leakage? (resonator stability)
  - How does the wave nature of the beam affect the resonator?
    - Gaussian beams, longitudinal and transverse modes
  - How can the resonator to control the beam profile?
  - How can we control and measure the output wavelength?
  - What types of beams can we produce?

Overview of physics in lasers

- Leak some out for the output beam:



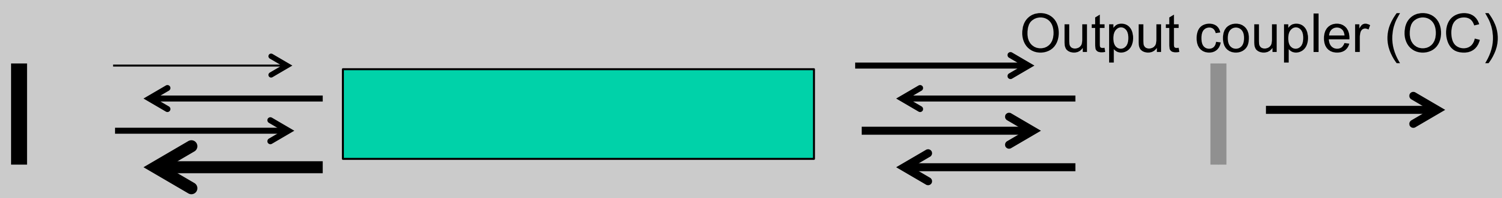
$$I_n = I_0\left(e^{2gL}\left(1-T\right)\right)^n \equiv I_0\left(e^{2gL}e^{-\gamma}\right)^n$$

Threshold: gain = loss

$2gL-\gamma$ $\qquad$ $I_n = I_0$

Overview of physics in lasers: system design, dynamics

- Leak some out for the output beam:



$$I_n = I_0\left(e^{2gL}\left(1-T\right)\right)^n \equiv I_0\left(e^{2gL}e^{-\gamma}\right)^n$$

Threshold: gain = loss

$2gL-\gamma$ $\qquad$ $I_n = I_0$

  - How do we design/optimize pumping system?
  - How is gain, energy extraction affected by gain distribution, beam profile, thermal effects?
  - How can we characterize the laser performance?
  - What happens away from steady state?
  - How do we get pulses out of the laser?

Simple 1D scalar wave equation

$$\frac{\partial^2}{\partial z^2}\psi(z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(z,t) = 0$$

- 2^{nd} order PDE
- Assume separable solution $\psi(z,t) = f\left(z\right)g\left(t\right)$

$$\frac{1}{f\left(z\right)}\frac{\partial^2}{\partial z^2}f(z) - \frac{1}{c^2}\frac{1}{g\left(t\right)}\frac{\partial^2}{\partial t^2}g\left(t\right) = 0$$

  - Each part is equal to a constant *A*

$$\frac{1}{f\left(z\right)}\frac{\partial^2}{\partial z^2}f(z) = A,\ \frac{1}{c^2}\frac{1}{g\left(t\right)}\frac{\partial^2}{\partial t^2}g\left(t\right) = A$$

$$f(z) = \cos\left(kz\right) \rightarrow -k^2 = A,\ g\left(t\right) = \cos\left(\omega\, t\right) \rightarrow -\omega^2\frac{1}{c^2} = A$$

$$\omega = \pm k\, c$$

Sin( ) also works as a second solution

Full solution of wave equation

- Full solution is a linear combination of both solutions

$$\psi(z,t) = f(z)g(t) = \left(A_1 \cos kz + A_2 \sin kz\right)\left(B_1 \cos \omega t + B_2 \sin \omega t\right)$$

- Too messy: use complex solution instead:

$$\psi(z,t) = f(z)g(t) = \left(A_1 e^{ikz} + A_2 e^{-ikz}\right)\left(B_1 e^{i\omega t} + B_2 e^{-i\omega t}\right)$$

$$\psi(z,t) = A_1 B_1 e^{i(kz+\omega t)} + A_2 B_2 e^{-i(kz+\omega t)} + A_1 B_2 e^{i(kz-\omega t)} + A_2 B_1 e^{-i(kz-\omega t)}$$

  – Constants are arbitrary: rewrite

$$\psi(z,t) = A_1 \cos\left(kz + \omega t + \phi_1\right) + A_2 \cos\left(kz - \omega t + \phi_2\right)$$

Interpretation of solutions

- Wave vector $$k = \frac{2\pi}{\lambda}$$
- Angular frequency $$\omega = 2\pi\nu$$
- Wave total phase: $$\Phi = kz - \omega t + \phi$$
  - "absolute phase": ϕ
  - Phase velocity: *c* $$\Phi = kz - k\,c\,t + \phi = k\left(z - c\,t\right) + \phi$$

$$\Phi = \text{constant when } z = ct$$

$$\psi(z,t) = A_1 \cos\left(kz + \omega t + \phi_1\right) + A_2 \cos\left(kz - \omega t + \phi_2\right)$$

Reverse (to -z) Forward (to +z)

Simple 3D scalar wave equation

$$\frac{\partial^2}{\partial x^2}\psi(x,y,z,t)+\frac{\partial^2}{\partial y^2}\psi(x,y,z,t)+\frac{\partial^2}{\partial z^2}\psi(x,y,z,t)-\frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\psi(x,y,z,t)=0$$

Refractive index changes velocity

- Still a 2nd order PDE
- Assume separable solution $\psi(z,t)=f_x\left(x\right)f_y\left(y\right)f_z\left(z\right)g\left(t\right)$

$$\psi(z,t)=$$
$$\left(A_{1x}e^{ik_x x}+A_{2x}e^{-ik_x x}\right)\left(A_{1y}e^{ik_y y}+A_{2y}e^{-ik_y y}\right)\left(A_{1z}e^{ik_z z}+A_{2z}e^{-ik_z z}\right)\left(B_1e^{i\omega t}+B_2e^{-i\omega t}\right)$$

$$\psi(z,t)=$$
$$A_1\cos\left(k_x x+k_y y+k_z z+\omega t+\phi_1\right)$$
$$+A_2\cos\left(k_x x+k_y y+k_z z-\omega t+\phi_2\right)$$

Wave vectors and the wave equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi(x,y,z,t)-\frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\psi(x,y,z,t)=0$$

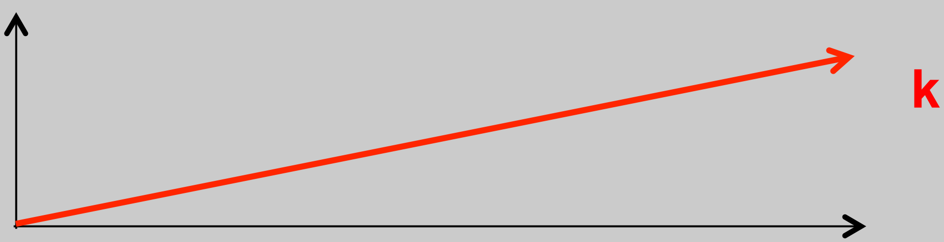
$$\boxed{\rightarrow \nabla^2\psi(\mathbf{r},t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\mathbf{r},t)=0}$$

$$\psi(z,t)=$$

$$A_1\cos\left(k_x x+k_y y+k_z z+\omega t+\phi_1\right)$$
$$+A_2\cos\left(k_x x+k_y y+k_z z-\omega t+\phi_2\right)$$

$$\boxed{\begin{aligned}&\rightarrow \psi(z,t)=A_1\cos\left(\mathbf{k}\cdot\mathbf{r}+\omega t+\phi_1\right)\\&+A_2\cos\left(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi_2\right)\end{aligned}}$$

k is a vector that defines the direction of the wave



$$\boxed{n^2\frac{\omega^2}{c^2}=k_x^2+k_y^2+k_z^2=\mathbf{k}\cdot\mathbf{k}}$$

Valid even in waveguides and resonators

Complex notation for waves

- Write cosine in terms of exponential

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(kz - \omega t + \phi) = \hat{\mathbf{x}} E_x \frac{1}{2}\left(e^{i(kz-\omega t+\phi)} + e^{-i(kz-\omega t+\phi)}\right)$$

  - Note E-field is a *real* quantity.
  - It is convenient to work with just one part
    - We will use $E_0 e^{+i(kz-\omega t)}$ $E_0 = \frac{1}{2} E_x e^{i\phi}$
    - Svelto: $e^{-i(kz-\omega t)}$
  - Then take the real part.
    - No factor of 2
    - In *nonlinear* optics, we have to explicitly include conjugate term

Example: linear resonator (1D)

- Boundary conditions: conducting ends (mirrors)

$$E_x\left(z=0,t\right)=0 \qquad E_x\left(z=L_z,t\right)=0$$

- Field is a superposition of +'ve and –'ve waves:

$$E_x\left(z,t\right)=A_+e^{i\left(k_z z-\omega t+\phi_+\right)}+A_-e^{i\left(-k_z z-\omega t+\phi_-\right)}$$

  – Absorb phase into complex amplitude

$$E_x\left(z,t\right)=\left(A_+e^{+ik_z z}+A_-e^{-ik_z z}\right)e^{-i\omega t}$$

  – Apply b.c. at z = 0

$$E_x\left(0,t\right)=0=\left(A_++A_-\right)e^{-i\omega t}\rightarrow A_+=-A_-$$

$$E_x\left(z,t\right)=A\sin k_z z\, e^{-i\omega t}$$

Quantization of frequency: longitudinal modes

- Apply b.c. at far end

$$E_x\left(L_z,t\right)=0=A\sin k_z L_z\, e^{-i\omega t} \qquad \rightarrow k_z L_z = l\pi \qquad l=1,2,3,\cdots$$

  - Relate to wavelength:

$$k_z=\frac{2\pi}{\lambda}=\frac{l\pi}{L_z}\rightarrow L_z=l\frac{\lambda}{2}$$

Integer number of half-wavelengths

  - Relate to allowed frequencies:

$$\frac{\omega_l}{c}=\frac{l\pi}{L_z}\rightarrow \nu_l=l\frac{c}{2L_z}$$

  - Equally spaced frequencies:

$$\Delta\nu=\frac{c}{2L_z}=\frac{1}{T_{RT}}$$

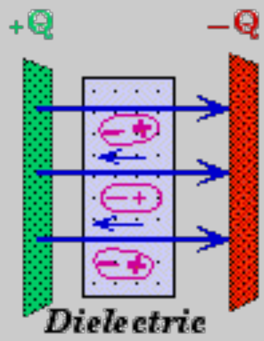
Frequency spacing = 1/ round trip time

Wave energy and intensity

- Both E and H fields have a corresponding energy density (J/m^3)
  - For static fields (e.g. in capacitors) the energy density can be calculated through the work done to set up the field

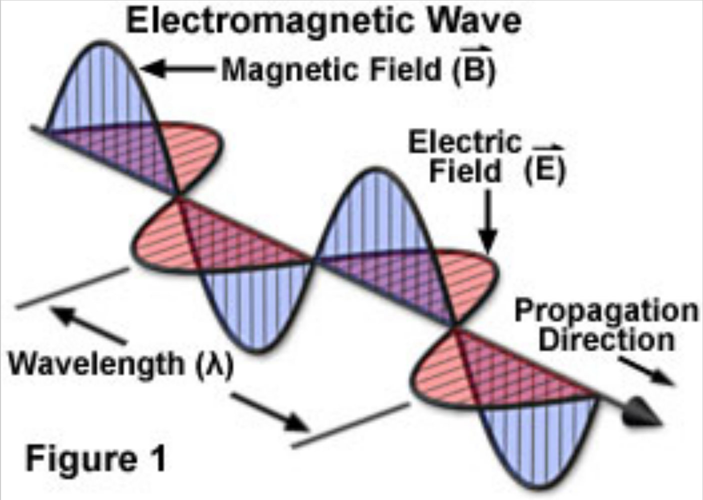
$$\rho = \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu H^2$$

  - Some work is required to polarize the medium
  - Energy is contained in both fields, but H field can be calculated from E field



H field from E field

- Maxwell equations relate E and H fields
- H field for a propagating wave is *in phase* with E-field



$$\mathbf{H} = \hat{\mathbf{y}} H_0 \cos\left(k_z z - \omega t\right)$$

$$= \hat{\mathbf{y}} \frac{k_z}{\omega \mu_0} E_0 \cos\left(k_z z - \omega t\right)$$

- Amplitudes are not independent

$$H_0 = \frac{n}{c\mu_0} E_0 = n\varepsilon_0 c E_0$$

- Note: field is polarized, two possible directions

Energy density in an EM wave

- The energy of the EM wave resides in both E and H fields
- Energy density (J/m^3)

$$\rho = \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu_0 H^2$$

$$H = n\varepsilon_0 cE$$

$$\varepsilon = \varepsilon_0 n^2$$

$$\rho = \tfrac{1}{2}\varepsilon_0 n^2 E^2 + \tfrac{1}{2}\mu_0 n^2 \varepsilon_0{}^2 c^2 E^2$$

$$\mu_0 \varepsilon_0 c^2 = 1$$

$$\rho = \varepsilon_0 n^2 E^2 = \varepsilon_0 n^2 E^2 \cos^2\left(k_z z - \omega t\right)$$

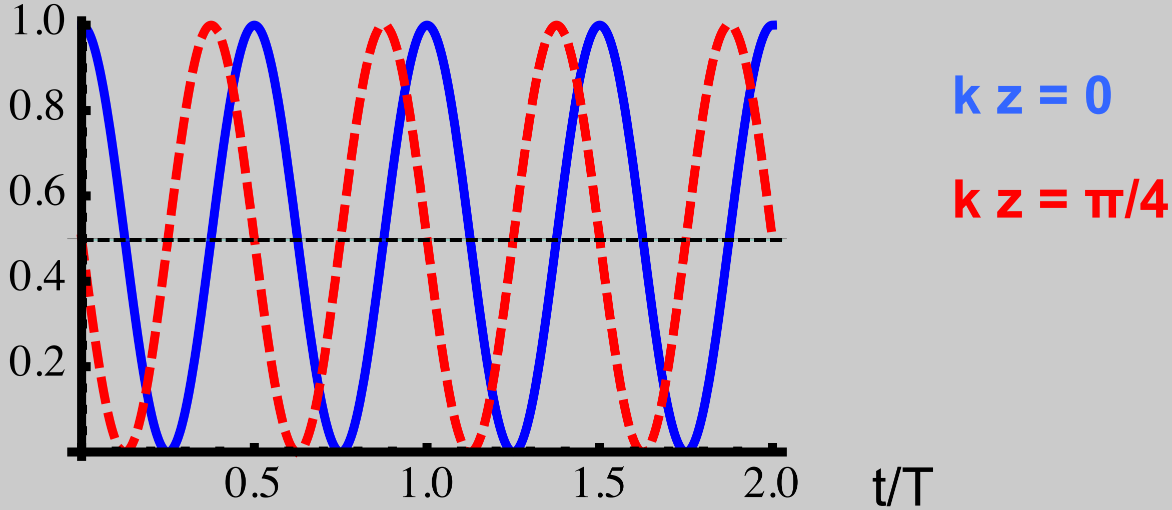
Equal energy in both components of wave

Cycle-averaged energy density

- Optical oscillations are faster than detectors
- Average over one cycle:

$$\langle \rho \rangle = \varepsilon_0 n^2 E_0^{\ 2} \frac{1}{T} \int_0^T \cos^2\left(k_z z - \omega t\right) dt$$

  – Graphically, we can see this should = ½



  – Regardless of position z

$$\boxed{\langle \rho \rangle = \frac{1}{2} \varepsilon_0 n^2 E_0^{\ 2}}$$

General 3D plane wave solution

- Assume separable function

$$\mathbf{E}(x,y,z,t) \sim f_1(x) f_2(y) f_3(z) g(t)$$

$$\vec{\nabla}^2 \mathbf{E}(z,t) = \frac{\partial^2}{\partial x^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial y^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial z^2}\mathbf{E}(z,t) = \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}(z,t)$$

- Solution takes the form:

$$\mathbf{E}(x,y,z,t) = \mathbf{E_0} e^{ik_x x} e^{ik_y y} e^{ik_z z} e^{-i\omega t} = \mathbf{E_0} e^{i\left(k_x x + k_y y + k_z z\right)} e^{-i\omega t}$$

$$\mathbf{E}(x,y,z,t) = \mathbf{E_0} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

  – Now k-vector can point in arbitrary direction

- With this solution in W.E.:

$$\boxed{n^2 \frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 = \mathbf{k}\cdot\mathbf{k}}$$

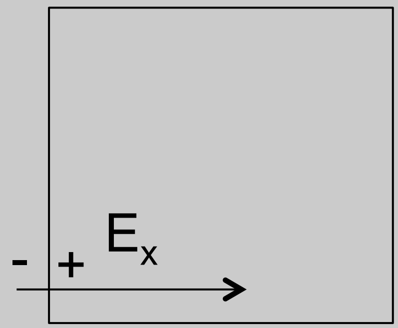
Valid even in waveguides and resonators

Closed box resonator: blackbody cavity

- Here we have a 3D pattern of standing waves
  - Exact boundary conditions aren't imp't, but for conducting walls:
    - E=0 where field is parallel to wall
    - Slope E=0 where field is perp to wall (charges can accumulate there)
  - Example standing wave solution:

$$E_x(x,y,z) = A_x \cos k_x x \sin k_y y \sin k_z z$$

    - Cos( ) function along field direction



  - Others:

$$E_y(x,y,z) = A_y \sin k_x x \cos k_y y \sin k_z z$$

$$E_z(x,y,z) = A_z \sin k_x x \sin k_y y \cos k_z z$$

Discrete wavevectors

- Discrete values of k:

$$k_x = \frac{l\pi}{L_x} \qquad k_y = \frac{m\pi}{L_y} \qquad k_z = \frac{n\pi}{L_z}$$

- With these solutions in the wave equation

$$\frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 = \mathbf{k}\cdot\mathbf{k}$$

2 allowed polarizations

  – k's are discrete, so there are discrete allowed frequencies:

$$\omega_{lmn} = c\sqrt{k_x^2 + k_y^2 + k_z^2} = c\sqrt{\left(\frac{l\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_z}\right)^2}$$

$$\nu_{lmn} = \frac{c}{2\pi}\sqrt{k_x^2 + k_y^2 + k_z^2} = c\sqrt{\left(\frac{l}{2L_x}\right)^2 + \left(\frac{m}{2L_y}\right)^2 + \left(\frac{n}{2L_z}\right)^2}$$

Field in equilibrium with walls: classical

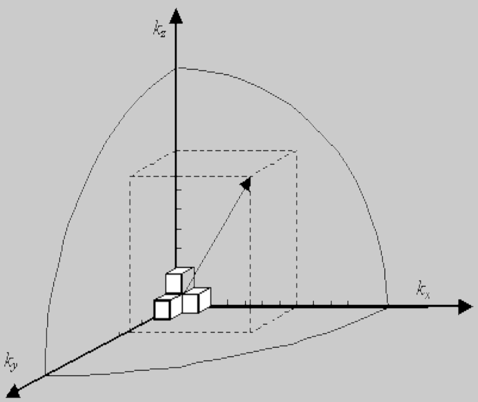
- Hold cavity walls at temperature T
- What is probability that a mode will be excited?
- Classical view (Boltzmann): $P(\mathcal{E}) \propto e^{-\mathcal{E}/kT}$
  - assume the amount of energy in each mode can take any value (continuous range) this is wrong!
  - average energy for each mode is

$$\langle \mathcal{E} \rangle = \frac{\int_0^\infty \mathcal{E}\, P(\mathcal{E})\, d\mathcal{E}}{\int_0^\infty P(\mathcal{E})\, d\mathcal{E}} = \frac{\int_0^\infty \mathcal{E}\, e^{-\mathcal{E}/kT}\, d\mathcal{E}}{\int_0^\infty e^{-\mathcal{E}/kT}\, d\mathcal{E}} = kT$$

  - Note: this is not kT/2 as in equipartition of K.E. There, integrate on velocity, which ranges – to +

Density of states

- For a given box size, there is a low frequency cutoff but no cutoff for high frequencies
- Near a given frequency, there will be a number of combinations of k's *l,m,n* for that frequency



$$N(k) = \#\text{pol states} \times \frac{\text{volume of k-space octant}}{\text{volume of unit k-space cell}}$$

$$= 2\frac{\frac{1}{8}(4/3)\pi k^3}{\frac{\pi}{L_x}\times\frac{\pi}{L_y}\times\frac{\pi}{L_z}} = \frac{k^3}{3\pi^2}V$$

Density of modes = density of states $\quad g(k)\,dk = \frac{1}{V}\frac{dN(k)}{dk}dk = \frac{k^2}{\pi^2}dk$

Other forms: $\quad g(\omega)\,d\omega = \frac{\omega^2}{\pi^2 c^3}d\omega \quad g(\nu)\,d\nu = 8\pi\frac{\nu^2}{c^3}d\nu$

Spectral energy density

- Generalize EM energy density to allow for spectral distribution

$$\rho(\nu)d\nu = \text{excitation energy per mode} \times \text{density of modes}$$

  – Total energy density: $\int \rho(\nu)d\nu$

  – Classical form:

$$\rho(\nu)d\nu = k_B T \, \frac{8\pi\nu^2}{c^3} d\nu$$

  – Problem: total energy is infinite!

- Planck: only allow quantized energies for each mode

$$\mathcal{E} = \left(n + \tfrac{1}{2}\right)h\nu$$ n = number of photons in each mode

  – Now get average energy/mode with sum, not integral

$$P_n = \frac{e^{-\mathcal{E}_n/k_B T}}{\sum_j e^{-\mathcal{E}_j/k_B T}}$$ Mean photon number: $\bar{n} = \sum_n n P_n$

Blackbody spectrum

- Mean number of photons per mode:

$$\bar{n} = \sum_j n P_n = 1 \Big/ \left(e^{h\nu/k_B T} - 1 \right)$$

- Spectral energy density of BB radiation:

$$\rho(\nu)\,d\nu = \text{avg \# photons per mode} \times h\nu \text{ per photon} \times \text{density of modes}$$

$$= \frac{1}{e^{h\nu/k_B T} - 1} h\nu \, g(\nu)\,d\nu = 8\pi \frac{\nu^2}{c^3} \frac{h\nu}{e^{h\nu/k_B T} - 1} d\nu$$