

1-30-08

Note Title

1/29/2008

we show that for QHO

$$E \geq \frac{\hbar\omega}{2}$$

Hence there is a lowest energy state: the ground state ψ_0 .

Now suppose we assert that

$$a\psi_0 = 0 \Rightarrow \frac{d\psi_0}{dx} + \frac{m\omega}{\hbar} x \psi_0 = 0$$

$$\Rightarrow \psi_0 = A e^{-\frac{m\omega}{2\hbar} x^2}$$

Claim: this is the ground state

proof: $H\psi_0 =$

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_0 + \frac{1}{2} m\omega^2 x^2 \psi_0$$

$$= A \left[-\frac{\hbar^2}{2m} \left(-\frac{m\omega}{\hbar} + \left(\frac{m\omega}{\hbar}\right)^2 x^2 \right) e^{-m\omega/2\hbar x^2} \right] \\ + A \frac{1}{2} m\omega^2 x^2 e^{-m\omega/2\hbar x^2}$$

$$= A \left\{ \frac{\hbar\omega}{2} - \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2 x^2 \right\} e^{-\frac{m\omega}{2\hbar} x^2}$$

$$= \frac{\hbar\omega}{2} A e^{-\frac{m\omega}{2\hbar} x^2} = \frac{\hbar\omega}{2} \psi_0$$

so $H \psi_0 = \frac{\hbar\omega}{2} \psi_0$

Thus ψ_0 is an ϵ -vector of H with ϵ -value $\frac{\hbar\omega}{2}$. since

$E \geq \frac{\hbar\omega}{2}$ this must be the

ground state.

Example $A_1 \psi_1 = a^+ \psi_0$

↑

normalization

$$a^+ \psi_0 = \frac{m\omega}{2\hbar} \left(x - \frac{i}{m\omega} (-i\hbar \frac{d}{dx}) \right) \psi_0$$

$$= \frac{m\omega}{2\hbar} \left[x - \frac{\hbar}{m\omega} \frac{d}{dx} \right] \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega}{2\hbar} x^2}$$

$$= \frac{m\omega}{2\hbar} \left[x + \frac{\hbar}{m\omega} \frac{m\omega}{\hbar} x \right] \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-m\omega/2\hbar x^2}$$

$$= \frac{m\omega}{\hbar} x \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-m\omega/2\hbar x^2}$$

$$= A_0 \frac{m\omega}{\hbar} x \psi_0 = \frac{\sqrt{2}}{\pi^{1/4}} \left(\frac{m\omega}{\hbar} \right)^{3/4} x e^{-\frac{(m\omega)}{2\hbar} x^2}$$

Simplify notation

$$e^{-m\omega/2\hbar x^2} = e^{-y^2}$$

$$y = \sqrt{\alpha} x$$

$$\alpha = \frac{m\omega}{\hbar}$$

$$\psi_0 = \left(\frac{\alpha}{\pi} \right)^{1/4} e^{-y^2/2}$$

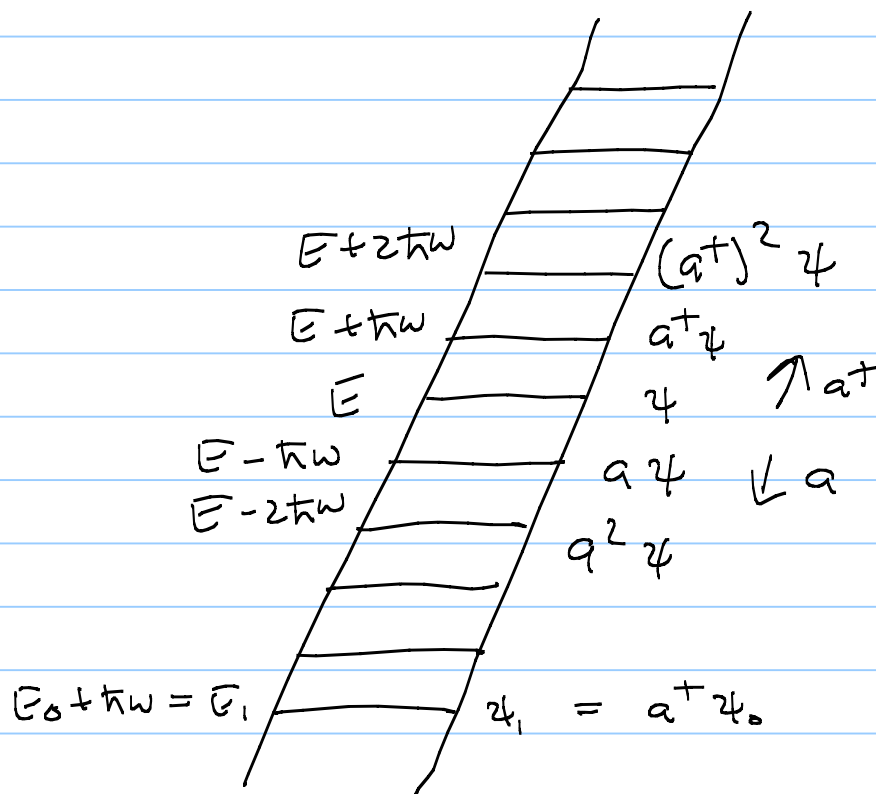
$$\frac{d}{dx} = \sqrt{\alpha} \frac{d}{dy}$$

$$a^+ = \sqrt{\frac{\alpha}{2}} \left(\frac{1}{\sqrt{\alpha}} y + \frac{1}{\alpha} \sqrt{\alpha} \frac{d}{dy} \right)$$

$$= \frac{1}{\sqrt{2}} \left[y + \frac{1}{\sqrt{\alpha}} \frac{d}{dy} \right]$$

.....

Ladder operators for QHO



All we need is on stationary state and we can compute all the rest by raising with a^+ or lowering with a .

$$H \psi_n = \hbar\omega \left(n + \frac{1}{2} \right) \psi_n \quad \text{QHO}$$

Review continued.

QHO. $\omega^2 = \frac{k}{m} \Rightarrow k = m\omega^2$

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

Be able to prove that $[x, p] = i\hbar$

Ladder operators

$$a = \frac{m\omega}{2\hbar} \left(x + \frac{i}{m\omega} p \right)$$

$$a^\dagger = \frac{m\omega}{2\hbar} \left(x - \frac{i}{m\omega} p \right)$$

Eg $a \psi_E$

Use $[H, a] = -\hbar\omega a$

i.e. $Ha - aH = -\hbar\omega a$

$$\Rightarrow H a \psi_E = \underbrace{a H \psi_E}_{= E \psi_E} - \hbar\omega a \psi_E$$

= $E \psi_E$ since it's
an eigenstate

$$\Rightarrow H a \psi_E = E a \psi_E - \hbar\omega a \psi_E$$

$$\Rightarrow H [a \psi_E] = (E - \hbar\omega) [a \psi_E]$$

So $a\psi_E$ is an Σ -state of H .
with Σ -value $E - \hbar\omega$: Lowering

Similarly using $[H, a^\dagger] = \hbar\omega a^\dagger$

$$\Rightarrow H [a^\dagger \psi_E] = (E + \hbar\omega) [a^\dagger \psi_E]$$

So $a^\dagger \psi_E$ is an Σ -state of H with
 Σ -value $E + \hbar\omega$: Raising.

Summary of what we know so far.

QM (non-relativistic anyway) is governed by TISE

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi$$

S. of Var. $\psi(x,t) = \psi(x)\phi(t)$

gives $\psi(x,t) = \psi(x) e^{-iEt/\hbar}$

analogous to normal modes in classical physics

Equation for $\psi(x)$ is

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + V(x)\psi = E\psi$$

or $\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \psi = E\psi$

Hamiltonian operator

$$H\psi = E\psi$$

So the spatial part of the S.E. is an ϵ -value ϵ -vector problem for H .

H is self-adjoint (Hermitian)

- So 1) H has real ϵ -values
2) a complete set of orthog. ϵ -vectors

$$H \psi_n = E_n \psi_n$$

$$\Psi_n(x, t) = \psi_n e^{-i E_n t / \hbar} \quad \text{solution to TISE}$$

But $\Psi_n(x, 0) = \psi_n(x)$ not general enough for arbitrary B.C. So we exploit linearity of S.E.

$$\Psi(x, t) = \sum C_n \psi_n(x) e^{-i E_n t / \hbar}$$

to calculate C_n we do

$$\Psi(x, 0) = \sum C_n \psi_n(x)$$

$$\int_{-\infty}^{\infty} \psi_m^*(x) \psi(x_0) dx = \sum_n C_n \underbrace{\int \psi_m^* \psi_n dx}_{\delta_{mn}} = C_m$$

$$C_m = \int_{-\infty}^{\infty} \psi_m^*(x) \psi(x_0) dx$$

E.g. for the ∞ -square well

$$C_m = \sqrt{\frac{2}{a}} \int_{-\infty}^{\infty} \sin\left(\frac{n\pi}{a}x\right) \psi(x_0) dx$$

$|C_n|^2$ tells you the probability that a measurement of the energy would yield a value E_n .

Hence the sum of the $|C_n|^2$ must be 1.

$$\sum_{n=1}^{\infty} |c_n|^2 = 1$$

Proof $1 = \int_{-\infty}^{\infty} |\psi(x,0)|^2 dx$

$$= \int_{-\infty}^{\infty} \left(\sum_m c_m \psi_m \right)^* \left(\sum_n c_n \psi_n \right) dx$$

$$= \int \sum_m \sum_n c_m^* c_n \psi_m^* \psi_n dx$$

$$= \sum_m \sum_n c_m^* c_n \underbrace{\int \psi_m^* \psi_n dx}_{\delta_{mn}}$$

$$= \sum c_m^* c_m = \sum |c_m|^2 = 1$$

$$\langle H \rangle = \int \bar{\Psi}(x,t) H \Psi(x,t) dx$$

$$\bar{\Psi}(x,t) = \sum_n c_n^* \psi_n(x) e^{-iE_n t/\hbar}$$

$$\langle H \rangle = \int \sum_n c_n^* \psi_n^* e^{iE_n t/\hbar} H \sum_m c_m \psi_m e^{-iE_m t/\hbar} dx$$

$$= \int \sum_n \sum_m C_n^* \psi_n^* e^{i(c)} C_m E_m \psi_m e^{-i(c)}$$

$$= \sum_n \sum_m C_n^* C_m E_m \int \psi_n^* \psi_m dx$$

cancel

$$\langle H \rangle = \sum_n E_n |C_n|^2$$

This is independent of time

manifestation of conservation
of energy.

we did example 2.2 last time
 ∞ square well with

$$\psi(x,0) = Ax(a-x) \quad 0 \leq x \leq a$$

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$$

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$$

we computed

$$C_n = \begin{cases} 0 & n \text{ even} \\ \frac{8\sqrt{15}}{(n\pi)^3} & n \text{ odd} \end{cases}$$

So let's verify:

$$|C_{11}|^2 = \left(\frac{8\sqrt{15}}{\pi^3}\right)^2 = .99855$$

So a measurement of Energy has
a 99.855% prob. of yielding
 $E_1 = \pi^2 \hbar^2 / 2ma^2$

less obvious

$$\sum_{n=1}^{\infty} |C_n|^2 = \left(\frac{8\sqrt{15}}{\pi^3}\right)^2 \sum_{n=1,3,5} \frac{1}{n^6} \equiv 1$$

Of course I wouldn't expect you to know that

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = 1 + \left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \dots = \frac{1}{1 - \frac{1}{3}}$$

but $\left(\frac{1}{3}\right)^6 = \frac{1}{15625}$ a very small number.

so to a good approx $\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n \approx 1 + \left(\frac{1}{3}\right)^6 = 1.00137$

Review continued.

