Day 13: Separation of variables in cylindrical coords

Laplace's eqn in cylindrical coords looks like:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

As usual, we'll cut ourselves a break and only consider "long" objects, so that there's no z-dependence in V and we can cut down to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Also as usual, hope for a solution of the form $V(r,\phi) = R(r)\Phi(\phi)$

Plug & crank to get

$$\Phi\frac{1}{r}\frac{d}{dr}\left(r\frac{d(R)}{dr}\right) + \frac{1}{r}R\frac{d^2\Phi}{d\phi^2} = 0$$

$$\Phi\left[\frac{1}{r}\frac{dR}{dr} + \frac{d^2R}{dr^2}\right] + \frac{1}{r}R\frac{d^2\Phi}{d\phi^2} = 0 \qquad \text{Multiply by } \frac{r}{\Phi R} \text{ to separate}$$

$$\frac{1}{R}\left[\frac{dR}{dr} + r\frac{d^2R}{dr^2}\right] + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

So $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = K_1^2$, $\frac{1}{R}\left[\frac{dR}{dr} + r\frac{d^2R}{dr^2}\right] = K_2^2$ with $K_1^2 + K_2^2 = 0$

The $\Phi$ solutions need to be periodic ($\phi$ coordinate goes from 0 to $2\pi$), so they're the sines/cosines:

$$\Phi(\phi) = C_n\cos(n\phi) + D_n\sin(n\phi)$$

Note the periodicity also forces $K_1$ to be integer, hence the $n$

That leaves (since $K_2^2 = n^2$)

$$\frac{dR}{dr} + r\frac{d^2R}{dr^2} - n^2R = 0$$

This is in our bag of semi-standard Diff Eqs, and the general solution is

$$R(r) = A_n r^n + B_n r^{-n}$$

When $n=0$ we get a special case:

$$r\frac{d^2R}{dr^2} + \frac{dR}{dr} = 0 \qquad \text{With solution} \quad A\ln r + B$$

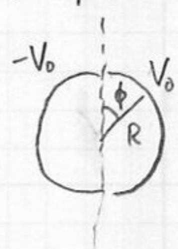
The full meal deal is

$$V(r,\phi) = A\ln(r) + B + \sum_{n=1}^{\infty} \left(A_n r^n + B_n r^{-n}\right)\left(C_n \cos n\phi + D \sin n\phi\right)$$

Zonal harmonics, or, as normal people would say, sines and cosines

Note that, as usual, you can often exclude big chunks of this based on symmetry or other considerations.

Example: Double hemi-cylinder

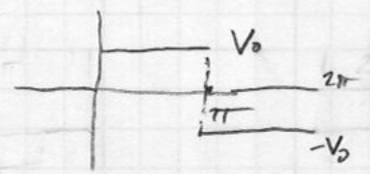



Conducting cylinder of radius R split into two halves held at $-V_0$ and $+V_0$. Lets hack away at V.

Ditch B. Poor B.

Ditch A. This object is net neutral, so as $r \to \infty$, we don't expect $V \to \infty$.

Now, the potential BC as I've drawn it is odd in $\phi$, so we need $\phi$-functions that respect that. Lose the cosines.




Here's a twist: $r^n$ blows up as $r \to \infty$ (which we don't think we want) and $r^{-n}$ blows up as $r \to 0$ (which we're really quite sure we don't want)

But we need some r-dependence, so we define V piecewise:

One V for $r \geq R$: $V(r,\phi)_{outside} = \sum_{n=1}^{\infty} B_n r^{-n} \sin(n\phi)$

Another V for $r \leq R$: $V(r,\phi)_{inside} = \sum_{n=1}^{\infty} A_n r^n \sin(n\phi)$

We'll need to make sure this is continuous across $r = R$. For that to happen we need

$$B_n R^{-n} = A_n R^n$$ For all $n$ (a rather harsh restriction)

Each of these is constant and equal to the other, so we can use a little shortcut gimmick and say

$$B_n R^{-n} = A_n R^n = C_n \Rightarrow B_n = C_n R^n$$

$$A_n = C_n R^{-n}$$

Leaving us with only one set of constants to fix:

$$V(r,\phi)_{out} = \sum_{n=1}^{\infty} c_n \left(\frac{R}{r}\right)^n \sin(n\phi)$$

$$V(r,\phi)_{in} = \sum_{n=1}^{\infty} c_n \left(\frac{r}{R}\right)^n \sin(n\phi)$$

Now we use Fourier's trick at $r=R$, where $V(R,\phi) = \sum_{n=1}^{\infty} c_n \sin(n\phi)$

$$\int_0^{2\pi} V(R,\phi)\sin(m\phi)\,d\phi = \sum_{n=1}^{\infty} c_n \underbrace{\int_0^{2\pi} \sin(n\phi)\sin(m\phi)\,d\phi}_{\pi\delta_{nm}} = c_m \cdot \pi$$

$V(R,\phi)$ is $V_0$ from $0$ to $\pi$, $-V_0$ from $\pi$ to $2\pi$

$$\int_0^{\pi} V_0 \sin(m\phi) + \int_{\pi}^{2\pi} -V_0 \sin(m\phi) = V_0 \cdot \left[ \frac{1}{m}\cos(m\phi)\Big|_0^{\pi} - \frac{1}{m}\cos(m\phi)\Big|_{\pi}^{2\pi} \right]$$

$$= V_0\left[\frac{1}{m}\cos(m\pi) - \frac{1}{m}\cos(0) - \frac{1}{m}\cos(2\pi m) + \frac{1}{m}\cos(m\pi)\right]$$

$$= V_0\left[\frac{2}{m}\cos(m\pi) - \frac{2}{m}\right]$$

$$= \frac{2V_0}{m}\left[\cos(m\pi) - 1\right]$$

For $m = 0, 2, 4,$ etc $\cos m\pi = 1$ and this zeroes out

For $m = 1, 3,$ etc $\cos(m\pi) = -1$ and we get $\frac{4V_0}{(2j+1)}$

$$\Rightarrow c_m = \frac{4V_0}{\pi(2j+1)}$$

Putting it together,

$$V(r,\phi)_{out} = \frac{4V_0}{\pi} \sum_{j=1}^{\infty} \frac{1}{(2j+1)} \left(\frac{R}{r}\right)^{(2j+1)} \sin((2j+1)\phi)$$

$$V(r,\phi)_{in} = \frac{4V_0}{\pi} \sum_{j=1}^{\infty} \left(\frac{1}{2j+1}\right) \left(\frac{r}{R}\right)^{(2j+1)} \sin((2j+1)\phi)$$

Note that I set up my coordinates slightly different than the book did, and so got a slightly different answer in $\phi$. It's worth pondering how these can be equivalent, since nothing changed physically.