

Fourier Transforms: Transform pairs, theorems in (x, y) and (β_X, β_Y) domains

Definitions and theorems (in Mathematica, use FourierParameters->{1,-1}):

Forward transform: $\mathfrak{F}\{g(x, y)\} \equiv G(\beta_X, \beta_Y) = \int_{-\infty}^{\infty} g(x, y) \exp[-i(\beta_X x + \beta_Y y)] dx dy$

Inverse transform: $\mathfrak{F}^{-1}\{G(\beta_X, \beta_Y)\} = g(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\beta_X, \beta_Y) \exp[i(\beta_X x + \beta_Y y)] d\beta_X d\beta_Y$

Shift Theorem: $\mathfrak{F}\{g(x - x_0)\} = \exp(-i\beta_X x_0) G(\beta_X)$ $\mathfrak{F}^{-1}\{G(\beta_X - \beta_{X0})\} = \exp(i\beta_{X0} x) g(x)$

Scale Theorem: $\mathfrak{F}\{g(ax)\} = \frac{1}{|a|} G(\beta_X / a)$ $\mathfrak{F}^{-1}\{G(b\beta_X)\} = \frac{1}{|b|} f(x/b)$

Conjugate: $\mathfrak{F}\{g^*(x)\} = G^*(-\beta_X)$

Inverse transform pair: $\mathfrak{F}\{G(x)\} = g(-\beta_X)$ $\mathfrak{F}^{-1}\{g(\beta_X)\} = G(-x)$

Convolution: $h(x) = f(x) \otimes g(x) = \int_{-\infty}^{\infty} f(x') g(x - x') dx'$

Convolution w/delta fcn: $\delta(x - x_0) \otimes g(x) = g(x - x_0)$

Convolution theorem:

$f(x) \otimes g(x) = \mathfrak{F}^{-1}\{F(\beta_X) G(\beta_X)\}$ $\mathfrak{F}\{f(x)g(x)\} = \frac{1}{2\pi} F(\beta_X) \otimes G(\beta_X)$

Parseval's theorem (conservation of energy): $\int |g(x)|^2 dx = \frac{1}{2\pi} \int |G(\beta_X)|^2 d\beta_X$

Transform pairs:

Delta functions:

$\mathfrak{F}\{\exp[\pm i\beta_{X0} x]\} = 2\pi \delta(\beta_X \pm \beta_{X0})$ $\mathfrak{F}^{-1}\{\exp[\pm i\beta_X x_0]\} = \delta(x \mp x_0)$

Gaussian:

$\mathfrak{F}\{\exp(-\pi x^2 / x_w^2)\} = x_w \exp(-\pi x_w^2 \beta_X^2)$

Rect function ($\text{rect}(u) = 1$ for $-1/2 < u < 1/2$):

$\mathfrak{F}\{\text{rect}(x/x_0)\} = x_0 \text{sinc}(\beta_X x_0 / 2)$

$\mathfrak{F}\{\text{sinc}(x/2x_0)\} = 2\pi x_0 \text{rect}(\beta_X x_0)$

Cosine function:

$\mathfrak{F}\{\cos(\beta_{X0} x)\} = \pi [\delta(\beta_X - \beta_{X0}) + \delta(\beta_X + \beta_{X0})]$

Array comb ($\text{comb}(x/x_0) \equiv \sum_{n=-\infty}^{\infty} \delta(x - nx_0)$):

$\mathfrak{F}\{\text{comb}(x/x_0)\} = 2\pi \text{comb}[\beta_X / (2\pi / x_0)]$

Circ function: ($\text{circ}(r/a) = 1$ for $r < a$)

$\mathfrak{F}\{\text{circ}(r/a)\} = \frac{2\pi a J_1(a\rho)}{\rho} = \pi a^2 \text{jinc}(a\rho)$,

where $\rho = \sqrt{\beta_X^2 + \beta_Y^2}$, and $\text{jinc}(x) = 2J_1(x)/x$