Fourier Transforms: Transform pairs, theorems in (x, y) and (β_X , β_Y) domains

Definitions and theorems (in Mathematica, use FourierParameters->{1,-1}:

Forward transform: $\Im\{g(x,y)\} \equiv G(\beta_X,\beta_Y) = \int_{-\infty}^{\infty} g(x,y)\exp[-i(\beta_X x + \beta_Y y)]dxdy$

Inverse transform: $\Im^{-1}\{G(f_X,f_Y)\} = g(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\beta_X,\beta_Y)\exp[i(\beta_X x + \beta_Y y)]d\beta_X d\beta_Y$

Shift Theorem: $\Im\{g(x-x_0)\} = \exp(-i\beta_X x_0)G(f_X)$ $\Im^{-1}\{G(\beta_X - \beta_{X0})\} = \exp(i\beta_{X0}x)g(x)$

Scale Theorem: $\Im\{g(ax)\} = \frac{1}{|a|}G(f_X / a)$ $\Im^{-1}\{G(bf_X)\} = \frac{1}{|b|}f(x/b)$

Conjugate: $\Im\{g^*(x)\} = G^*(-\beta_X)$

Inverse transform pair: $\Im\{G(x)\} = g(-\beta_X)$ $\Im^{-1}\{g(\beta_X)\} = G(-x)$

Convolution: $h(x) = f(x)\otimes g(x) = \int_{-\infty}^{\infty} f(x')g(x-x')dx'$

Convolution w/delta fcn: $\delta(x-x_0)\otimes g(x) = g(x-x_0)$

Convolution theorem:

$f(x)\otimes g(x) = \Im^{-1}\{F(\beta_X)G(\beta_X)\}$ $\Im\{f(x)g(x)\} = F(\beta_X)\otimes G(\beta_X)$

Parseval's theorem (conservation of energy): $\int |g(x)|^2 dx = \frac{1}{2\pi}\int |G(\beta_X)|^2 d\beta_X$

<u>Transform pairs:</u>

Delta functions:

$\Im\{\exp[\pm i\beta_{X0}x]\} = 2\pi\delta(\beta_X \pm \beta_{X0})$ $\Im^{-1}\{\exp[\pm i\beta_X x_0]\} = \delta(x \mp x_0)$

Gaussian: $\Im\{\exp(-\pi x^2 / x_w^2)\} = x_w \exp(-\pi x_w^2 f_X^2)$

Rect function (rect(u) = 1 for – ½ < u < ½): $\Im\{\text{rect}(x/x_0)\} = x_0\,\text{sinc}(\beta_X x_0 /2)$

$\Im\{\text{sinc}(x/2x_0)\} = 2\pi x_0 rect(\beta_X x_0)$

Cosine function: $\Im\{\cos(\beta_{X0}x)\} = \pi[\delta(\beta_X - \beta_{X0}) + \delta(\beta_X + \beta_{X0})]$

Array $comb(x/x_0) \equiv \sum_{n=-\infty}^{\infty}\delta(x - nx_0)$: $\Im\{comb(x/x_0)\} = 2\pi\, comb[\beta_x/(2\pi/x_0)]$

Circ function: (circ(r/a) = 1 for r<a) $\Im\{circ(r/a)\} = \frac{2\pi a J_1(a\rho)}{\rho} = \pi a^2 jinc(a\rho)$,

where $\rho = \sqrt{\beta_X^2 + \beta_Y^2}$, and $jinc(x) = 2J_1(x)/x$