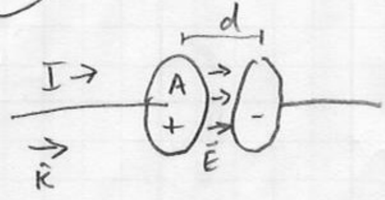
① (From Griffiths 8.2)

a)



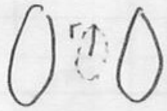
The E in between two plates goes like $\sigma/\varepsilon_0 = \frac{Q}{A\varepsilon_0}$

Letting $t=0$ be the time when $Q=0$, we get

$$\boxed{\vec{E} = \frac{It}{A\varepsilon_0}\hat{k}}$$

where the fact that I flows into the positive plate lets us know that I and $\bar{E}$ go in the same direction

$\bar{B}$ comes from applying Ampere-Maxwell:



$\oint \bar{B}\cdot d\bar{l} = \mu_0(I+I_D)$ In between, $I=0$, $I_D = \varepsilon_0 \frac{dE}{dt}\cdot$(area enclosed)

$$\Rightarrow B\cdot 2\cancel{\pi} r\cancel{t} = \mu_0 \cdot \cancel{\varepsilon_0}\cdot \frac{I}{A\cancel{\varepsilon_0}}\cdot \cancel{\pi} r^{\cancel{2}}$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 I r}{2A}\hat{\phi}}$$

where the sign comes from the right hand rule (note that $\oint B\cdot d\bar{l} = +\mu_0 I_D$ as opposed to $\oint \vec{E}\cdot d\bar{l} = -\frac{d\Phi}{dt}$)

b) The energy density $u_{EM} = \frac{1}{2\mu_0}B^2 + \frac{1}{2}\varepsilon_0 E^2$, so

$$u_{EM} = \frac{I^2t^2}{2A^2\varepsilon_0} + \frac{\mu_0 I^2 r^2}{8A^2} = \boxed{\frac{I^2}{A^2}\left(\frac{t^2}{2\varepsilon_0} + \frac{\mu_0 r^2}{8}\right)}$$

$\hat{r}\ \hat{\phi}\ \hat{k}\ \hat{r}\ \hat{\phi}\ \hat{k}$

And $\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{It}{A\varepsilon_0}\cdot\frac{Ir}{2A}\cdot(\hat{k}\times\hat{\phi})$ With $\hat{k}\times\hat{\phi} = -\hat{r}$

$$\boxed{\vec{S} = -\frac{I^2 r t}{2A^2\varepsilon_0}\hat{r}}$$

This says the field energy flow is <u>radially inward</u>: energy is moving from the space around the capacitor into the space between the plates.

Poynting's theorem says $\nabla \cdot \vec{S} = -\frac{\partial u_K}{\partial t} - \frac{\partial u_{em}}{\partial t}$, and here $\frac{\partial u_K}{\partial t} = 0$

since there are no charges in the region of interest.

We'll calculate each other term:

$$\nabla \cdot \vec{S} = \frac{1}{r}\frac{\partial}{\partial r}\left(r S_r\right) = -\frac{1}{r}\frac{d}{dr}\left(\frac{I^2 r^2 t}{2A^2\varepsilon_0}\right) = -\frac{I^2 t}{A^2 \varepsilon_0}$$

$$-\frac{\partial u_{em}}{\partial t} = -\frac{d}{dt}\frac{I^2t^2}{2\varepsilon_0 A^2} = -\frac{I^2 t}{A^2\varepsilon_0}$$

So Poynting's theorem holds.

c) We're going to calculate the energy stored in the shaded region.

$$U(t) = \int u_{em}\, dV = \int_0^d \int_0^{2\pi} \int_0^{R \text{ (radius of plates)}} \left(\frac{I^2t^2}{2\varepsilon_0 A^2} + \frac{\mu_0 I^2 r^2}{8A^2}\right) r\,dr\,d\phi\,dz$$

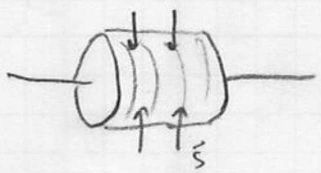
$$= 2\pi d \left[\int_0^R \frac{I^2t^2}{2\varepsilon_0 A^2} r\,dr + \int_0^R \frac{\mu_0 I^2 r^3}{8A^2}\,dr\right]$$

$$= 2\pi d \left[\frac{I^2t^2R^2}{4\varepsilon_0 A^2} + \frac{\mu_0 I^2 R^4}{32A^4}\right]$$

And the time derivative of this gives us the power being delivered:

$$\frac{dU}{dt} = \cancel{2}\pi d \cdot \frac{\cancel{2}I^2tR^2}{\cancel{4}\varepsilon_0 A^2} = \frac{I^2 t d(\pi R^2)}{\varepsilon_0 A^2} = \frac{I^2 t d}{\varepsilon_0 A} \quad (\text{since } A = \pi R^2)$$

Now we directly calculate the power inflow using $\bar{S}$:



Power is crossing that cylindrical surface, so we need $\vec{S}$ at $r=R$

$$\vec{S}\Big|_{r=R} = -\frac{I^2Rt}{2A^2\varepsilon_0}\hat{r}$$

$\vec{S}|_{r=R}$ is constant all along the surface of integration, so

$$\int \bar{S}\cdot d\bar{A}' = \int \frac{I^2Rt}{2A^2\varepsilon_0}\,dA' = \frac{I^2Rt}{\cancel{2}A^2\varepsilon_0}\cdot\cancel{2}\pi R d$$

not the same A

$$= \frac{I^2td}{A\varepsilon_0}\cdot\cancel{\frac{\pi R^2}{A}}$$

$$= \frac{I^2td}{A\varepsilon_0}$$

Which matches $\frac{dU}{dt}$, as it should. You'll notice I was a bit loose with the signage on the last part - there's some sign ambiguity on open surface flux integrals, so don't sweat it too much.

It's also kind of notable that the only dynamic term comes from E and not B. Normally it wouldn't be like that, but we have an E that is linear in t, resulting in a time-independent B.

2) (From Pollack & Stump 11.17)

We're given $\vec{E} = \frac{E_0}{\sqrt{2}}(\hat{k} - \hat{\imath})\sin(ky - \omega t)$

a) $\vec{B}$ is such that $\vec{E}\cdot\vec{B} = 0$ (they're perpendicular) and $|\vec{E}| = c|\vec{B}|$. Unfortunately many vectors are $\perp$ to $\vec{E}$, so we need a tighter constraint. Let's work from $\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$

$$\nabla\times\vec{E} = \frac{\partial E_z}{\partial y}\hat{\imath} - \frac{\partial E_x}{\partial y}\hat{k} = \frac{E_0 k\cos(ky-\omega t)}{\sqrt{2}}(\hat{\imath} + \hat{k}) = -\frac{\partial\vec{B}}{\partial t}$$

Integrate w.r.t. time and get

$$\boxed{\vec{B} = \frac{E_0 k}{\omega\sqrt{2}}\sin(ky - \omega t)(+\hat{\imath} + \hat{k})}$$

Since $(-\hat{\imath} - \hat{k})\cdot(-\hat{\imath} + \hat{k}) = 0$ and since $c = k/\omega$, everything checks out.

b) We get $\vec{S}$ from $\frac{\vec{E}\times\vec{B}}{\mu_0}$. I'll gather up all the scalar terms:

$$\vec{S} = \frac{1}{\mu_0}\frac{E_0 k}{\sqrt{2}}\cdot\frac{E_0 k}{\omega\sqrt{2}}\sin^2(ky - \omega t)(\hat{k} - \hat{\imath})\times(+\hat{\imath} + \hat{k})$$

$$\vec{S} = \frac{E_0^2 k}{2\mu_0\omega}\sin^2(ky - \omega t)(+2\hat{\jmath}) = \boxed{\frac{E_0^2 k}{\mu_0\omega}\sin^2(ky - \omega t)(+\hat{\jmath})}$$

The $\sin^2$ dependence is consistent with the fact that $|\vec{S}| = I$, the intensity, which should be $\propto$ to E or B squared.

The $\hat{\jmath}$ propagation direction is consistent with the fact that this is supposed to be a transverse wave.