10-3-07

Note Title 10/3/2007

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} i & 2 \\ 3 & -i \end{bmatrix}^+ = \overline{\begin{bmatrix} i & 3 \\ 2 & -i \end{bmatrix}} = \begin{bmatrix} -i & 3 \\ 2 & i \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2} \\ i \end{bmatrix}^+ = \overline{\begin{bmatrix} \sqrt{2} & i \end{bmatrix}} = \begin{bmatrix} \sqrt{2} & -i \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2} \\ i \end{bmatrix}^+ \begin{bmatrix} \sqrt{2} \\ i \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -i \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ i \end{bmatrix} = 3$$

Real	Complex
A^T	$A^{\dagger} = \overline{A^T}$
symmetric	Hermitian
$A = A^T$	$A^{\dagger} = A$
orthogonal	Unitary
$A^T A = A A^T = I$	$A^{\dagger} A = A A^{\dagger} = I$

Both symmetric & Hermitian matrices have real E-values

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Let $\vec{r}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\vec{r}_2 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

be 2 E-vectors of

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ with } \lambda_1, \lambda_2$$

$$\boxed{\begin{aligned} A\vec{r}_1 &= \lambda_1 \vec{r}_1 \\ A\vec{r}_2 &= \lambda_2 \vec{r}_2 \end{aligned}}$$

Now

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 y_1 \\ \lambda_1 x_2 & \lambda_2 y_2 \end{bmatrix}$$

$\uparrow$ $\uparrow$

$\vec{r}_1$ $\vec{r}_2$

$$= \begin{bmatrix} \lambda_1 \vec{r}_1 & \lambda_2 \vec{r}_2 \end{bmatrix}$$

on the other han

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 & a_{11}y_1 + a_{12}y_2 \\ a_{21}x_1 + a_{22}x_2 & a_{21}y_1 + a_{22}y_2 \end{bmatrix}$$

$$\qquad\qquad\quad \vec{r}_1 \quad \vec{r}_2 \qquad\qquad \underbrace{\qquad\qquad}_{\vec{A}\cdot\vec{r}_1} \qquad \underbrace{\qquad\qquad}_{A\cdot\vec{r}_2}$$

Thus

$$A\begin{bmatrix} \vec{r}_1 & \vec{r}_2 \end{bmatrix} = \begin{bmatrix} \vec{r}_1 & \vec{r}_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

is completely equivalent to

$$A\vec{r}_1 = \lambda_1 \vec{r}_1$$
$$A\vec{r}_2 = \lambda_2 \vec{r}_2$$

But now we see that if

$$C \equiv \begin{bmatrix} \vec{r}_1 & \vec{r}_2 \end{bmatrix} \text{ is invertible}$$

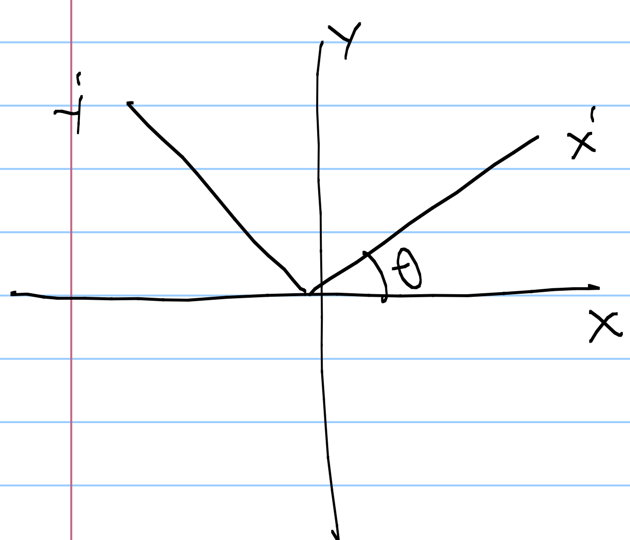
Then with $D \equiv \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

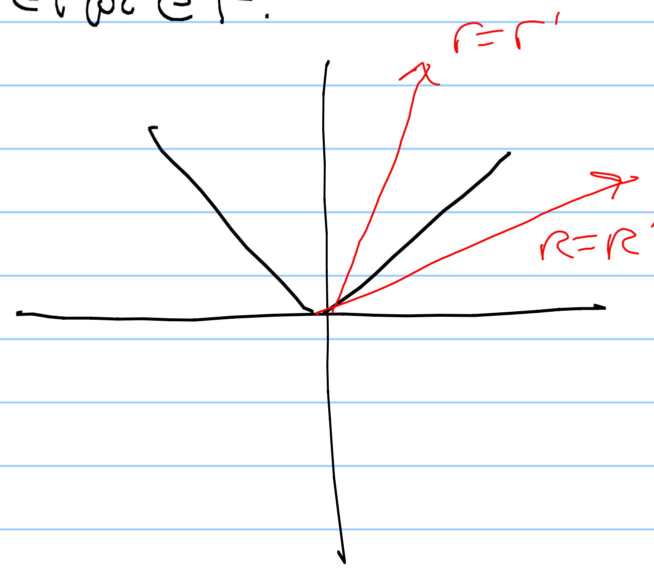
$$AC = CD \Rightarrow$$

$$C^{-1}AC = D$$

diagonal.

Geometrical Interpret.





rotation

$x = x'\cos\theta - y'\sin\theta$

$y = x'\sin\theta + y'\cos\theta$

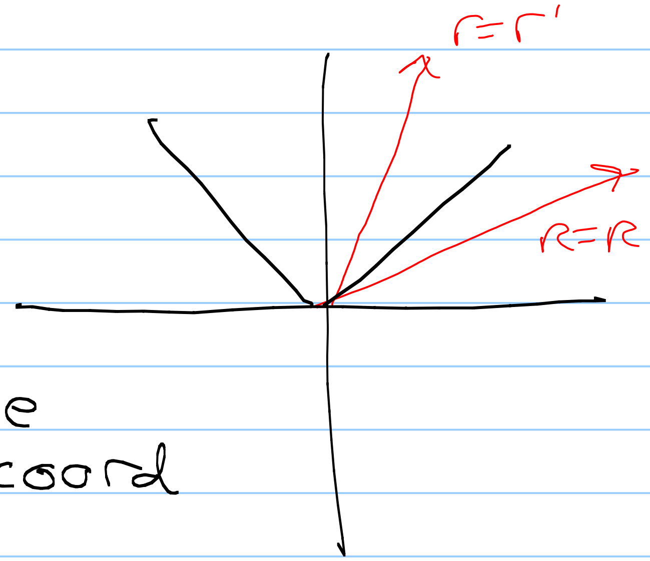
or

$x' = x\cos\theta + y\sin\theta$

$y' = -x\sin\theta + y\cos\theta$

i.e. $\vec{r} = C\vec{r}\,'$

now $\vec{R}, \vec{R}'$
$\vec{r}, \vec{r}\,'$



represent the same vector in 2 coord systems.

$$\vec{R} = C\vec{R}'$$

Consider the transf. in the unprimed coordinates that takes $\vec{R} \to \vec{r}$ call it M

$$\vec{R} = M\vec{r}$$

in the primed coordinates this becomes

$$CR' = MC\vec{r}\,'$$

$$\Rightarrow \quad R' = C^{-1}MC\,\vec{r}\,'$$

So the transformation represented by M in the unprimed coordinates becomes $C^{-1} M C$ in the primed,

$$\vec{R} = M \vec{r}$$

$$\Downarrow$$

$$\vec{R}' = C^{-1} M C \vec{r}'$$

Diagonalizing a Matrix M amounts to transforming coordinates to one in which the transformation is diagonal.

Ex $\qquad A = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$

Char. polynomial : $\text{Det}\,|A - \lambda I| = 0$

$\Rightarrow \quad (5-\lambda)^2 - 1 = 0$

$\Rightarrow \quad 5 - \lambda = \pm 1$

$\Rightarrow \quad \boxed{\lambda = 5 \pm 1}$

now find Σ-vectors

$$\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 6\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} 5x + y &= 6x \\ x + 5y &= 6y \end{aligned}\right\} x = y$$

so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} 5x + y &= 4x \\ x + 5y &= 4y \end{aligned}\right\} x = -y$$

so $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Let $Q = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right)$

$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ Let $\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} = M$

we showed that

$MQ = Q\Lambda$ $\Lambda = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$

Q is orthogonal ($Q^T Q = I$)

so $M = Q \Lambda Q^T$ Lets check

$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is symmetric

but any invertible matrix satisfies

$$\boxed{C^{-1} M C = M}$$

since M is symm, we are free to choose C to be orthog.

$$Q \Lambda Q^T = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 6 & 6 \\ 4 & -4 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 10 & 2 \\ 2 & 10 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$$

Any matrix with N distinct E-values can be diagonalized

Any Symmetric matrix can be diagonalized by an orthogonal matrix

Any Hermitian matrix can be diagonalized by a unitary matrix what can go wrong

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ char. poly:

$$(-\lambda)(-\lambda) = 0$$

$$\lambda^2 = 0$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow y = 0$$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an ε-vector

<u>for Both ε-values!</u>

$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ char polynom

$$(3-\lambda)^2 = 0$$

$$3-\lambda = 0 \implies \begin{matrix} \lambda_1 = 3 \\ \lambda_2 = 3 \end{matrix}$$

repeated ε- values

$$A\vec{r}_1 = 3\vec{r}_1 \qquad A\vec{r}_2 = 3\vec{r}_2$$

$$\vec{r}_1 \propto \vec{r}_2 \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

if N ε-vectors correspond to N different ε-values then the ε-vectors are linearly indep.

A set of vectors is linearly independent if none of them can be written as a linear comb. of finitely many other vectors in the set.

$$\underbrace{\overbrace{\begin{bmatrix}0\\0\\1\end{bmatrix} \quad \begin{bmatrix}0\\2\\-2\end{bmatrix} \quad \begin{bmatrix}1\\-2\\1\end{bmatrix}}^{\text{indep.}} \quad \begin{bmatrix}4\\2\\3\end{bmatrix}}_{\text{depend.}}$$

$$9\begin{bmatrix}0\\0\\1\end{bmatrix} + 5\begin{bmatrix}0\\2\\-2\end{bmatrix} + 4\begin{bmatrix}1\\-2\\1\end{bmatrix} = \begin{bmatrix}4\\2\\3\end{bmatrix}$$