PHGN 341 Exam1 Solutions (max: 60 pts)

1.

(a) $Z = \left(\sum_s e^{-\varepsilon_s/\tau}\right)^N = Z_1^N \qquad Z_1 = e^{3J/\tau} + 3e^{-J/\tau}$

(b) $U = \tau^2 \frac{\partial \ln Z}{\partial \tau} = N\tau^2 \frac{\left(\frac{\partial Z_1}{\partial \tau}\right)}{Z_1} = \frac{-3JN\left(e^{3J/\tau} - e^{-J/\tau}\right)}{\left(e^{3J/\tau}+3e^{-J/\tau}\right)}$ or $U = F + \tau\sigma$

(c) $F = -\tau \ln Z = -\tau N \ln\left(e^{3J/\tau} + 3e^{-J/\tau}\right) = -N\tau \ln Z_1$

(d) $\tau\sigma = U - F$ or $\sigma = -\frac{\partial F}{\partial \tau} = N \ln\left(e^{3J/\tau} + 3e^{-J/\tau}\right) + N\tau\left(\frac{-3J}{\tau^2}\right)\left(\frac{e^{3J/\tau} - e^{-J/\tau}}{e^{3J/\tau}+3e^{-J/\tau}}\right)$

(e) $\tau \to 0 \quad \sigma \to N\frac{3J}{\tau} - N\frac{3J}{\tau} = 0$

$U \to -3JN$ (= N times lowest single particle energy $-3J$)

2. (a) $\mu = \tau \ln\left(\frac{n_g}{n_Q}\right)$

(b) $\mu_{ad} = \mu$ because the two systems are in equilibrium (diffusive + thermal)

(c) $P(1,-I) = \frac{e^{(I+\mu)/\tau}}{\mathcal{Z}} \qquad \mathcal{Z} = \underset{\substack{N=0\\ \varepsilon=0}}{1} + \underset{\substack{N=1\\ \varepsilon=-I}}{e^{(I+\mu)/\tau}}$

(d) $N_{ad} = N_s P(1,-I) \left(\text{or } \lambda \frac{\partial}{\partial\lambda} \ln \mathcal{Z}^{N_s} = N_{ad}\right) = N_s \frac{\lambda e^{I/\tau}}{\mathcal{Z}}$

(e) $N_{ad} \approx N_s \lambda e^{I/\tau} = N_s \frac{n_g}{n_Q} e^{I/\tau}$

3. (a) $\tau d\sigma = \tau\left(\frac{\partial\sigma}{\partial U}\right)_l dU + \tau\left(\frac{\partial\sigma}{\partial l}\right)_U dl = dU - f\,dl \Rightarrow \left(\frac{\partial\sigma}{\partial l}\right)_U = -\frac{f}{\tau}$

(b) s and $-s$ give same $l = 2a|s|$ so $g(N,l) = 2g(N,s) = g(N,s) + g(N,-s)$

(c) $\left(\frac{\partial\sigma}{\partial l}\right)_U = \frac{-N}{2L} \ln\left(\frac{L+l}{L-l}\right) = -\frac{f}{\tau} \qquad (L = Na)$

(d) $l = L \tanh\left(\frac{fa}{\tau}\right)$ As τ increases, rubber band shrinks (try it!)