

Lecture 15

Note Title

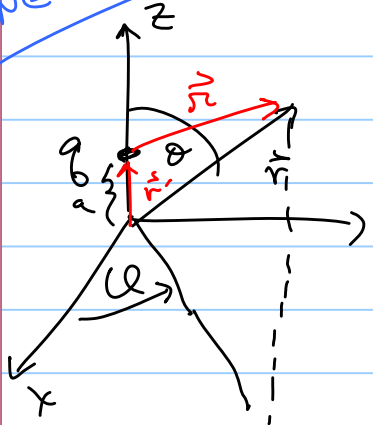
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Two types of electrostatics problems

- Boundary value problem (sep. variables)
- Summation problem

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho d\tau}{r} \quad \text{or} \quad \vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho \hat{r} d\tau}{r^2}$$

ONE CHARGE



$$V(r, \theta) = \frac{1}{4\pi\epsilon_0} \frac{q}{r'} = \frac{q}{4\pi\epsilon_0} \frac{1}{(r^2 + a^2 - 2ar \cos \theta)^{1/2}}$$

Assume $r > a$ and expand in small parameter $\frac{a}{r} - \frac{2a}{r} \cos \theta = \epsilon$
 \hookrightarrow Taylor or binomial expansion

$$V(r, \theta) \propto \frac{1}{r} = \frac{1}{r(1+\epsilon)^{1/2}} = \frac{1}{r} \left[1 - \frac{1}{2} \left(\frac{a^2}{r^2} - \frac{2a}{r} \cos \theta \right) + \frac{3}{8} \left(\frac{a^2}{r^2} - \frac{2a}{r} \cos \theta \right)^2 + \dots \right]$$

Now arrange in powers of $\frac{a}{r} \ll 1$; $\frac{1}{r} = \frac{1}{r} \left[1 + \left(\frac{a}{r} \right) \cos \theta + \left(\frac{a}{r} \right)^2 \left[\frac{3 \cos^2 \theta - 1}{2} \right] + \dots \right]$

$$\frac{1}{r} = \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{a}{r} \right)^l P_l(\cos \theta) \quad r > a$$

Note that for $a \ll r$ V is due to a point charge even though q is not at $\vec{r}' = 0$. Note that for $a \ll r$ V is a complicated function $\neq \frac{q}{r}$ the series converges slowly.

So why would we want to do this complicated expansion? It turns out we will use only the 2nd term in the expansion, since the other higher order terms will be negligible in the following chapters.

Ex: Axial multipole expansion

assume $\rho(\vec{r}') = \delta(x') \delta(y') \lambda(z')$ ← either continuous or delta functions

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}') d^3r'}{r} = \frac{1}{4\pi\epsilon_0} \iiint \delta(x') dx' \delta(y') dy' \frac{\lambda(z') dz'}{(r^2 + z'^2 - 2z'r \cos\theta)^{1/2}}$$

$$= \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(z')}{r} \sum_{l=0}^{\infty} \left(\frac{z'}{r}\right)^l P_l(\cos\theta) dz' = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{M_l P_l(\cos\theta)}{r^{l+1}}$$

where $M_l = \int \lambda(z') (z')^l dz'$ are the axial multipole moments!

$M_0 = \int \lambda(z') dz'$ is the monopole moment = Q_{total}

M_1 is the dipole moment. Note $V \propto \frac{1}{r^2}$ since the dipole contains a $+$ & $-$ charge.

example: $\lambda(z') = q_0 \delta(z' - \frac{a}{2}) - q_0 \delta(z' + \frac{a}{2})$

$$M_l = \int q_0 \delta(z' - \frac{a}{2}) (z')^l dz' - \int q_0 \delta(z' + \frac{a}{2}) (z')^l dz' = \begin{cases} 0 & l=0 \\ (\frac{a^l}{2}) 2q_0 & l=1,3,5,\dots \end{cases}$$