Day 29: Motional EMF & Faraday's Law I

"Mr. Faraday, of what use of your discoveries?"

"Madam, of what use is a baby?" - Michael Faraday

We're now moving into full-blown electrodynamics, where there will almost always be some time-dependence somewhere.
We need to start with a little vocabulary:

EMF: electromotive force; a measure of how much work a system is doing on a charge. Has units of volts, or J/C (energy/charge)

Physically very much like voltage, though "voltage" or "electric potential" tend to be used in situations where a scalar potential can be defined.

Since work is $\int \vec{F} \cdot d\vec{l}$ and the Lorentz force law is $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$, EMF will be of the form $\int \vec{F}/q \cdot d\vec{l} = \int (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{l}$

Generally we focus on either electric or magnetic forces.

We've seen Faraday's Law written two different ways in Phys 200:

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

$$EMF = -\frac{d\Phi}{dt}$$

And we're going to add a third:

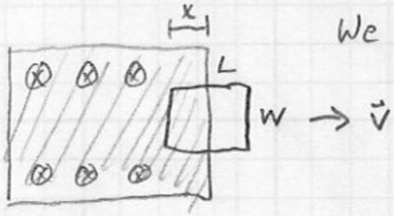
$$\vec{\nabla} \times \vec{E} = -\frac{dB}{dt}$$

(Clicker question)

Let's work through a few different situations and see what kind of insight we can obtain.

EMF from a static B-field

Let's suppose we have a B-field that occupies some region of space. It's uniform, with magnitude B, into the page. B is zero everywhere else. (This field is actually unphysical - do you see why?)



We also have a wire loop with dimensions W + L partly in the field. We're pulling it out with some speed v. x is the length of the loop still in the field.

The charge carriers in the wire (let's call them positive for convenience) feel an EMF due to the Lorentz force:

$$\mathcal{E}MF = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

Now, the right hand rule shows us that only charges in the leftmost wire segment are acted on (only there is $\vec{v} \times \vec{B} \parallel$ to $d\vec{l}$)

$$\Rightarrow \quad \mathcal{E}MF = vBW$$

Let's look at $\frac{d\Phi}{dt}$ also. $\Phi = \int \vec{B} \cdot d\vec{A}$, or in this case, BWx

Thus $\frac{d\Phi}{dt} = \frac{d}{dt}(BWx) = BWv$

So we find that $$\boxed{\mathcal{E}MF = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l} = -\frac{d\Phi}{dt}} \quad (1)$$

(What's that sign about?)

You'll notice there is <u>no $\vec{E}$ anywhere</u> in the problem.

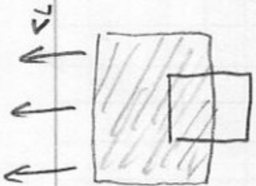
This turns out to be a general result for <u>any</u> moving loop in a B-field. You can prove it using $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{F} = q\vec{v} \times \vec{B}$.

To emphasize for future use: $$\boxed{\vec{\nabla} \cdot \vec{B} = 0 \ + \ \vec{F} = q\vec{v} \times \vec{B} \quad \Rightarrow \quad \mathcal{E}MF = -\frac{d\Phi}{dt}}$$

With no E, and being derivable from existing principles, this is <u>not</u> Faraday's Law (not when the EMF comes from a purely magnetic force. But it is quite closely related.

Electromagnetic Induction + Faraday's (actual) Law

Now, what of the reciprocal situation where the wire loop is stationary and the region occupied by the field is moving, thus providing a time varying $\vec{B}$?



We cannot derive what happens from what we currently know. We need a brand new, empirically obtained law: Faraday's Law in differential form:

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

This law tells us that E-fields have another source besides charges: time-varying magnetic fields

This law will let us figure out what happens in any situation where $\vec{B}$ is changing, thus inducing a curly EMF such that

$$\oint \vec{E} \cdot d\vec{l} \neq 0$$

And so we come full circle.

We have to proceed carefully, though. Many people (and books) screw up what follows as we try to get Faraday's Law into integral form. First, integrate both sides over some <u>open</u> surface:

$$\int (\nabla \times \vec{E}) \cdot d\vec{A} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

Stokes's thm on left

$$\oint \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

Right hand side is a spatial integral only, so we can pull out $\partial/\partial t$ and make it a total derivative

$$\boxed{\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}}$$

That seemed simple enough. And it's true that $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$ and $\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$ are equivalent <u>if and only if</u> B is the <u>only</u> thing with a time dependence.

What happens if we go back to the first situation, where $\vec{B}$ isn't changing but the loop is moving? Start with:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Integrate both sides and apply Stokes on left:

$$\oint \vec{E}\cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

Now we <u>don't</u> pull out $\frac{\partial}{\partial t}$

<u>Why not?</u>

Ok, so what we have here is that $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ and the above <u>are</u> equivalent with no restrictions whatsoever. But we can't necessarily get that $\partial/\partial t$ out if the area is changing, because the <u>bounds</u> of the integral are time dependent!

(2)

$$\oint \vec{E}\cdot d\vec{l} = -\int_{s(t)} \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

Note that this does give a consistent result in the case of the moving loop, since $\partial\vec{B}/\partial t = 0$ and $\vec{E} = 0$ there.

But can we do better? What can we do with that $\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$ term? Let's look at

$$-\frac{d}{dt}\int_{s(t)} \vec{B}\cdot d\vec{A}$$

and see if we can relate it.

You take the derivative of functions whose bounds have variables using the Leibniz rules. You've seen one in 1-D:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = \frac{db}{dt}f(b,t) - \frac{da}{dt}f(a,t) + \int_a^b \frac{\partial}{\partial t}f(x,t)\,dx$$

It's got a vaguely product/chain rule flavor to it, where to apply the derivative completely we need to hit the b-endpoint, the a-endpoint, and the integrand. This generalizes to vector fields instead of scalar functions, but not cleanly. Letting $\vec{B}$ be our vector field, the 3-D Leibniz rule gives:

$$\frac{d}{dt}\int B(\vec{x},t)\cdot d\vec{A} = \int\left(\frac{\partial \vec{B}}{\partial t} + (\vec{\nabla}\cdot\vec{B})\vec{v}\right)\cdot d\vec{A} - \oint(\vec{v}\times\vec{B})\cdot d\vec{l}$$

Where $\vec{v}$ comes from the time derivative of the spatial coordinates defining the area (so, for example, the velocity of the wire loop).

$\vec{\nabla}\cdot\vec{B} = 0$, so we can find that

$$-\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A} = -\oint(\vec{v}\times\vec{B})\cdot d\vec{l} - \frac{d}{dt}\int \vec{B}(\vec{x},t)\cdot d\vec{A}$$

Substitute this into ② to get, for the case of the moving loop,

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A} - \oint (\vec{v}\times\vec{B})\cdot dl \qquad \text{or}$$

$$\oint (\vec{E} + \vec{v}\times\vec{B})\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$$

Now, interpret this one of two ways.

I) $\vec{E}=0$ in the moving loop scenario, so we recover ①. This is rather nice.

II) We can have $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$ be true in <u>both</u> reference frames if we define $\vec{E}' = \vec{E} + (\vec{v}\times\vec{B})$

WE JUST GOT THE RULE FOR HOW E-FIELDS TRANSFORM BETWEEN FRAMES!

Well, mostly. A proper relativistic derivation gives us γ's and stuff, but for $v \ll c$, $\vec{E}' = \vec{E} + (\vec{v}\times\vec{B})$

So we get a sniff, just a hint that the goofy structure of magnetism (like all the cross-product terms) comes from the physical principle that laws (like $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$) should look the same in any inertial reference frame and purely mathematical operations like the Leibniz rules.

Daaaaamn!