CHAPTER II

ELECTROSTATICS OF DIELECTRICS

§6. The electric field in dielectrics

WE SHALL now go on to consider a static electric field in another class of substances, namely dielectrics. The fundamental property of dielectrics is that a steady current cannot flow in them. Hence the static electric field need not be zero, as in conductors, and we have to derive the equations which describe this field. One equation is obtained by averaging equation (1.3), and is again

$$\mathbf{curl}\,\mathbf{E} = 0. \tag{6.1}$$

A second equation is obtained by averaging the equation $\mathrm{div}\,\mathbf{e} = 4\pi\rho$:

$$\mathrm{div}\,\mathbf{E} = 4\pi\bar{\rho}. \tag{6.2}$$

Let us suppose that no charges are brought into the dielectric from outside, which is the most usual and important case. Then the total charge in the volume of the dielectric is zero; even if it is placed in an electric field we have $\int \bar{\rho}\,dV = 0$. This integral equation, which must be valid for a body of any shape, means that the average charge density can be written as the divergence of a certain vector, which is usually denoted by $-\mathbf{P}$:

$$\bar{\rho} = -\mathrm{div}\,\mathbf{P}, \tag{6.3}$$

while outside the body $\mathbf{P} = 0$. For, on integrating over the volume bounded by a surface which encloses the body but nowhere enters it, we find $\int \bar{\rho}\,dV = -\int \mathrm{div}\,\mathbf{P}\,dV = -\oint \mathbf{P}\cdot d\mathbf{f} = 0$. $\mathbf{P}$ is called the *dielectric polarization*, or simply the *polarization*, of the body. A dielectric in which $\mathbf{P}$ differs from zero is said to be *polarized*. The vector $\mathbf{P}$ determines not only the volume charge density (6.3), but also the density σ of the charges on the surface of the polarized dielectric. If we integrate formula (6.3) over an element of volume lying between two neighbouring unit areas, one on each side of the dielectric surface, we have, since $\mathbf{P} = 0$ on the outer area (cf. the derivation of formula (1.9)),

$$\sigma = P_n, \tag{6.4}$$

where P_n is the component of the vector $\mathbf{P}$ along the outward normal to the surface.

To see the physical significance of the quantity $\mathbf{P}$ itself, let us consider the total dipole moment of all the charges within the dielectric; unlike the total charge, the total dipole moment need not be zero. By definition, it is the integral $\int \mathbf{r}\bar{\rho}\,dV$. Substituting $\bar{\rho}$ from (6.3) and again integrating over a volume which includes the whole body we have

$$\int \mathbf{r}\bar{\rho}\,dV = -\int \mathbf{r}\,\mathrm{div}\,\mathbf{P}\,dV = -\oint \mathbf{r}(d\mathbf{f}\cdot\mathbf{P}) + \int (\mathbf{P}\cdot\mathbf{grad})\mathbf{r}\,dV.$$

The integral over the surface is zero, and in the second term we have $(\mathbf{P}\cdot\mathbf{grad})\mathbf{r} = \mathbf{P}$, so that

$$\int \mathbf{r}\bar{\rho}\,dV = \int \mathbf{P}\,dV. \tag{6.5}$$

Thus the polarization vector is the dipole moment (or *electric moment*) per unit volume of the dielectric.†

Substituting (6.3) in (6.2), we obtain the second equation of the electrostatic field in the form

$$\mathrm{div}\,\mathbf{D} = 0, \tag{6.6}$$

where we have introduced a quantity $\mathbf{D}$ defined by

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \tag{6.7}$$

called the *electric induction*. The equation (6.6) has been derived by averaging the density of charges in the dielectric. If, however, charges not belonging to the dielectric are brought in from outside (we shall call these *extraneous charges*), then their density must be added to the right-hand side of equation (6.6):

$$\mathrm{div}\,\mathbf{D} = 4\pi\rho_{ex}. \tag{6.8}$$

On the surface of separation between two different dielectrics, certain boundary conditions must be satisfied. One of these follows from the equation $\mathbf{curl}\,\mathbf{E} = 0$. If the surface of separation is uniform as regards physical properties,‡ this condition requires the continuity of the tangential component of the field:

$$\mathbf{E}_{t1} = \mathbf{E}_{t2}; \tag{6.9}$$

cf. the derivation of the condition (1.7). The second condition follows from the equation $\mathrm{div}\,\mathbf{D} = 0$, and requires the continuity of the normal component of the induction:

$$D_{n1} = D_{n2}. \tag{6.10}$$

For a discontinuity in the normal component $D_n = D_z$ would involve an infinity of the derivative $\partial D_z/\partial z$, and therefore of $\mathrm{div}\,\mathbf{D}$.

At a boundary between a dielectric and a conductor, $\mathbf{E}_t = 0$, and the condition on the normal component is obtained from (6.8):

$$\mathbf{E}_t = 0, \quad D_n = 4\pi\sigma_{ex}, \tag{6.11}$$

where σ_{ex} is the charge density on the surface of the conductor; cf. (1.8), (1.9).

§7. The permittivity

In order that equations (6.1) and (6.6) should form a complete set of equations determining the electrostatic field, they must be supplemented by a relation between the induction $\mathbf{D}$ and the field $\mathbf{E}$. In the great majority of cases this relation may be supposed linear. It corresponds to the first terms in an expansion of $\mathbf{D}$ in powers of $\mathbf{E}$, and its correctness is due to the smallness of the external electric fields in comparison with the internal molecular fields.

The linear relation between $\mathbf{D}$ and $\mathbf{E}$ is especially simple in the most important case, that

† It should be noticed that the relation (6.3) inside the dielectric and the condition $\mathbf{P} = 0$ outside do not in themselves determine $\mathbf{P}$ uniquely; inside the dielectric we could add to $\mathbf{P}$ any vector of the form $\mathbf{curl}\,\mathbf{f}$. The exact form of $\mathbf{P}$ can be completely determined only by establishing its connection with the dipole moment.

‡ That is, as regards composition of the adjoining media, temperature, etc. If the dielectric is a crystal, the surface must be a crystallographic plane.

of an isotropic dielectric. It is evident that, in an isotropic dielectric, the vectors **D** and **E** must be in the same direction. The linear relation between them is therefore a simple proportionality:†

$$\mathbf{D} = \varepsilon \mathbf{E}. \tag{7.1}$$

The coefficient ε is the *permittivity* or *dielectric permeability* or *dielectric constant* of the substance and is a function of its thermodynamic state.

As well as the induction, the polarization also is proportional to the field:

$$\mathbf{P} = \kappa \mathbf{E} \equiv (\varepsilon - 1)\mathbf{E}/4\pi. \tag{7.2}$$

The quantity κ is called the *polarization coefficient* of the substance, or its *dielectric susceptibility*. Later (§14) we shall show that the permittivity always exceeds unity; the polarizability, accordingly, is always positive. The polarizability of a rarefied medium (a gas) may be regarded as proportional to its density.

The boundary conditions (6.9) and (6.10) on the surface separating two isotropic dielectrics become

$$\mathbf{E}_{t1} = \mathbf{E}_{t2}, \quad \varepsilon_1 E_{n1} = \varepsilon_2 E_{n2}. \tag{7.3}$$

Thus the normal component of the field is discontinuous, changing in inverse proportion to the permittivity of the medium.

In a homogeneous dielectric, ε = constant, and then it follows from div **D** = 0 that div **P** = 0. By the definition (6.3) this means that the volume charge density in such a body is zero (but the surface density (6.4) is in general not zero). On the other hand, in an inhomogeneous dielectric we have a non-zero volume charge density

$$\bar{\rho} = -\operatorname{div}\mathbf{P} = -\operatorname{div}\frac{\varepsilon-1}{4\pi\varepsilon}\mathbf{D} = -\frac{1}{4\pi}\mathbf{D}\cdot\mathbf{grad}\frac{\varepsilon-1}{\varepsilon} = -\frac{1}{4\pi\varepsilon}\mathbf{E}\cdot\mathbf{grad}\,\varepsilon.$$

If we introduce the electric field potential by $\mathbf{E} = -\mathbf{grad}\,\phi$, then equation (6.1) is automatically satisfied, and the equation div **D** = div ε**E** = 0 gives

$$\operatorname{div}(\varepsilon\,\mathbf{grad}\,\phi) = 0. \tag{7.4}$$

This equation becomes the ordinary Laplace's equation only in a homogeneous dielectric medium. The boundary conditions (7.3) can be rewritten as the following conditions on the potential:

$$\left.\begin{aligned} \phi_1 &= \phi_2, \\ \varepsilon_1 \partial\phi_1/\partial n &= \varepsilon_2 \partial\phi_2/\partial n; \end{aligned}\right\} \tag{7.5}$$

the continuity of the tangential derivatives of the potential is equivalent to the continuity of ϕ itself.

In a dielectric medium which is piecewise homogeneous, equation (7.4) reduces in each homogeneous region to Laplace's equation $\triangle\phi = 0$, so that the permittivity appears in the solution of the problem only through the conditions (7.5). These conditions, however,

† This relation, which assumes that **D** and **E** vanish simultaneously, is, strictly speaking, valid only in dielectrics which are homogeneous as regards physical properties (composition, temperature, etc.). In inhomogeneous bodies **D** may be non-zero even when **E** = 0, and is determined by the gradients of thermodynamic quantities which vary through the body. The corresponding terms, however, are very small, and we shall use the relation (7.1) in what follows, even for inhomogeneous bodies.

involve only the ratio of the permittivities of two adjoining media. In particular, the solution of an electrostatic problem for a dielectric body with permittivity ε_2, surrounded by a medium with permittivity ε_1, is the same as for a body with permittivity $\varepsilon_2/\varepsilon_1$, surrounded by a vacuum.

Let us consider how the results obtained in Chapter I for the electrostatic field of conductors will be modified if these conductors are not in a vacuum but in a homogeneous and isotropic dielectric medium. In both cases the potential distribution satisfies the equation $\triangle\phi = 0$, with the boundary condition that ϕ is constant on the surface of the conductor, and the only difference is that, instead of $E_n = -\partial\phi/\partial n = 4\pi\sigma$, we have

$$D_n = -\varepsilon\,\partial\phi/\partial n = 4\pi\sigma, \tag{7.6}$$

giving the relation between the potential and the surface charge. Hence it is clear that the solution of the problem of the field of a charged conductor in a vacuum gives the solution of the same problem with a dielectric in place of the vacuum if we make the formal substitution $\phi \to \varepsilon\phi$, $e \to e$ or $\phi \to \phi$, $e \to e/\varepsilon$. For given charges on the conductors, the potential and the field are reduced by a factor ε in comparison with their values in a vacuum. This reduction in the field can be explained as the result of a partial "screening" of the charge on the conductor by the surface charges on the adjoining polarized dielectric. If, on the other hand, the potentials of the conductors are maintained, then the field is unchanged but the charges are increased by a factor ε.†

Finally, it may be noted that in electrostatics we may formally regard a conductor (uncharged) as a body of infinite permittivity, in the sense that its effect on an external electric field is the same as that of a dielectric (of the same form) as $\varepsilon \to \infty$. For, since the boundary condition on the induction **D** is finite, **D** must remain finite in the body even for $\varepsilon \to \infty$. This means that $\mathbf{E} \to 0$, in accordance with the properties of conductors.

PROBLEMS

PROBLEM 1. Determine the field due to a point charge e at a distance h from a plane boundary separating two different dielectric media.

SOLUTION. Let O be the position of the charge e in medium 1, and O' its image in the plane of separation, situated in medium 2 (Fig. 11, p. 38). We shall seek the field in medium 1 in the form of the field of two point charges, e and a fictitious charge e' at O' (cf. the method of images, §3): $\phi_1 = e/\varepsilon_1 r + e'/\varepsilon_1 r'$, where r and r' are the distances from O and O' respectively. In medium 2 we seek the field as that of a fictitious charge e'' at O: $\phi_2 = e''/\varepsilon_2 r$. On the boundary plane ($r = r'$) the conditions (7.5) must hold, leading to the equations $e - e' = e''$, $(e + e')/\varepsilon_1 = e''/\varepsilon_2$, whence

$$e' = e(\varepsilon_1 - \varepsilon_2)/(\varepsilon_1 + \varepsilon_2), \qquad e'' = 2\varepsilon_2 e/(\varepsilon_1 + \varepsilon_2). \tag{1}$$

For $\varepsilon_2 \to \infty$ we have $e' = -e$, $\phi_2 = 0$, i.e. the result obtained in §3 for the field of a point charge near a conducting plane.

The force acting on the charge e (the *image force*) is

$$F = \frac{ee'}{(2h)^2\varepsilon_1} = \left(\frac{e}{2h}\right)^2 \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1(\varepsilon_1 + \varepsilon_2)};$$

$F > 0$ corresponds to repulsion.

PROBLEM 2. The same as Problem 1, but for an infinite charged straight wire parallel to a plane boundary surface at a distance h.

† From this it follows, in particular, that when a capacitor is filled with a dielectic its capacitance increases by a factor ε.

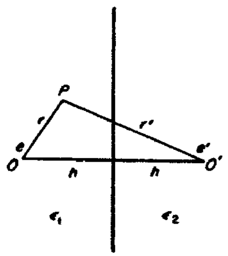

FIG. 11

SOLUTION. As in Problem 1, except that the field potentials in the two media are $\phi_1 = -(2e/\varepsilon_1)\log r - (2e'/\varepsilon_1)\log r'$, $\phi_2 = -(2e''/\varepsilon_2)\log r$, where e, e', e'' are the charges per unit length of the wire and of its images, and r, r' are the distances in a plane perpendicular to the wire. The same expressions (1) are obtained for e', e'', and the force on unit length of the wire is $F = 2ee'/2h\varepsilon_1 = e^2(\varepsilon_1-\varepsilon_2)/h\varepsilon_1(\varepsilon_1+\varepsilon_2)$.

PROBLEM 3. Determine the field due to an infinite charged straight wire in a medium with permittivity ε_1, lying parallel to a cylinder with radius a and permittivity ε_2, at a distance $b\,(> a)$ from its axis.†

SOLUTION. We seek the field in medium 1 as that produced in a homogeneous dielectric (with ε_1) by the actual wire (passing through O in Fig. 12), with charge e per unit length, and two fictitious wires with charges e' and $-e'$ per unit length, passing through A and O' respectively. The point A is at a distance a^2/b from the axis of the cylinder. Then, for all points on the circumference, the distances r and r' from O and A are in a constant ratio $r'/r = a/b$, and so it is possible to satisfy the boundary conditions on this circumference. In medium 2 we seek the field as that produced in a homogeneous medium (with ε_2) by a fictitious charge e'' on the wire passing through O.

The boundary conditions on the surface of separation are conveniently formulated in terms of the potential ϕ ($\mathbf{E} = -\mathbf{grad}\,\phi$) and the vector potential $\mathbf{A}$ (cf. §3), defined by $\mathbf{D} = \mathbf{curl}\,\mathbf{A}$ (in accordance with the equation $\operatorname{div}\mathbf{D} = 0$). In a two-dimensional problem, $\mathbf{A}$ is in the z-direction (perpendicular to the plane of the figure). The conditions of continuity for the tangential components of $\mathbf{E}$ and the normal component of $\mathbf{D}$ are equivalent to $\phi_1 = \phi_2$, $A_1 = A_2$.

For the field of a charged wire we have in polar coordinates r, θ the equation $\phi = -(2e/\varepsilon)\log r + \text{constant}$, $A = 2e\theta + \text{constant}$; cf. (3.18). Hence the boundary conditions are

$$\frac{2}{\varepsilon_1}(-e\log r - e'\log r' + e'\log a) = -\frac{2e''}{\varepsilon_2}\log r + \text{constant},$$

$$2[e\theta + e'\theta' - e'(\theta+\theta')] = 2e''\theta,$$

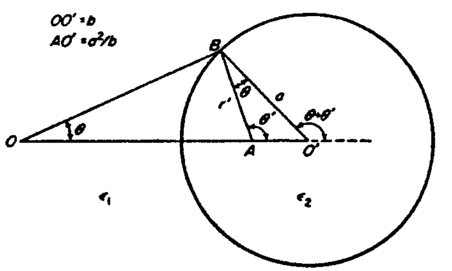

FIG. 12

† The corresponding problem of a point charge near a dielectric sphere cannot be solved in closed form.

where the angles are as shown in Fig. 12, and we have used the fact that $OO'B$ and $BO'A$ are similar triangles. Hence $\varepsilon_2(e+e') = \varepsilon_1 e''$, $e - e' = e''$, and the expressions for e' and e'' are again formulae (1) of Problem 1.

The force acting on unit length of the charged wire is parallel to OO', and is

$$F = eE = \frac{2ee'}{\varepsilon_1}\left(\frac{1}{OA} - \frac{1}{OO'}\right) = \frac{2e^2(\varepsilon_1-\varepsilon_2)a^2}{\varepsilon_1(\varepsilon_1+\varepsilon_2)b(b^2-a^2)};$$

$F > 0$ corresponds to repulsion. In the limit $a, b \to \infty$, $b - a \to h$, this gives the result in Problem 1.

PROBLEM 4. The same as Problem 3, but for the case where the wire is inside a cylinder with permittivity ε_2 $(b < a)$.

SOLUTION. We seek the field in medium 2 as that due to the actual wire, with charge e per unit length (O in Fig. 13), and a fictitious wire with charge e' per unit length passing through A, which is now outside the cylinder. In medium 1 we seek the field as that of wires with charges e'' and $e - e''$ passing through O and O' respectively. By the same method as in the preceding problem we find $e' = -e(\varepsilon_1-\varepsilon_2)/(\varepsilon_1+\varepsilon_2)$, $e'' = 2\varepsilon_1 e/(\varepsilon_1+\varepsilon_2)$. For $\varepsilon_2 > \varepsilon_1$ the wire is repelled from the surface of the cylinder by a force

$$F = \frac{2ee'}{\varepsilon_2}\frac{1}{OA} = \frac{2e^2(\varepsilon_2-\varepsilon_1)b}{\varepsilon_2(\varepsilon_1+\varepsilon_2)(a^2-b^2)}.$$

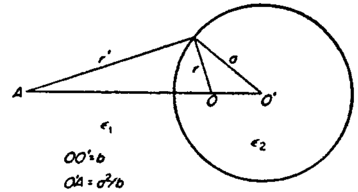

FIG. 13

PROBLEM 5. Show that the field potential $\phi_A(\mathbf{r}_B)$ at a point $\mathbf{r}_B$ in an arbitrary inhomogeneous dielectric medium, due to a point charge e at $\mathbf{r}_A$, is equal to the potential $\phi_B(\mathbf{r}_A)$ at $\mathbf{r}_A$ due to the same charge at $\mathbf{r}_B$.

SOLUTION. The potentials $\phi_A(\mathbf{r})$ and $\phi_B(\mathbf{r})$ satisfy the equations

$$\operatorname{div}(\varepsilon\,\mathbf{grad}\,\phi_A) = -4\pi e\delta(\mathbf{r}-\mathbf{r}_A), \quad \operatorname{div}(\varepsilon\,\mathbf{grad}\,\phi_B) = -4\pi e\delta(\mathbf{r}-\mathbf{r}_B).$$

Multiplying the first by ϕ_B and the second by ϕ_A and subtracting, we have

$$\operatorname{div}(\phi_B\varepsilon\,\mathbf{grad}\,\phi_A) - \operatorname{div}(\phi_A\varepsilon\,\mathbf{grad}\,\phi_B) = -4\pi e\delta(\mathbf{r}-\mathbf{r}_A)\phi_B(\mathbf{r}) + 4\pi e\delta(\mathbf{r}-\mathbf{r}_B)\phi_A(\mathbf{r}).$$

Integration of this equation over all space gives the required relation:

$$\phi_A(\mathbf{r}_B) = \phi_B(\mathbf{r}_A).$$

§8. A dielectric ellipsoid

The polarization of a dielectric ellipsoid in a uniform external electric field has some unusual properties which render this example particularly interesting.

Let us consider first a simple special case, that of a dielectric sphere in an external field $\mathfrak{E}$. We denote its permittivity by $\varepsilon^{(i)}$, and that of the medium surrounding it by $\varepsilon^{(e)}$. We take the origin of spherical polar coordinates at the centre of the sphere, and the direction of $\mathfrak{E}$ as the axis from which the polar angle θ is measured, and seek the field potential outside the