PHGN 462 Homework 9

1) (Read *all* of the below)

This problem is based on Pollack and Stump 15.3, and does a decent job of putting a nice, happy bow on all of classical electromagnetism.

Starting from Maxwell's equations with sources $\rho(\vec{x},t)$ and $\vec{J}(\vec{x},t)$, show that:

$$-\nabla^2\vec{E} + \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = -\frac{1}{\varepsilon_0}\nabla\rho - \mu_0\frac{\partial\vec{J}}{\partial t}$$

$$-\nabla^2\vec{B} + \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = \mu_0\nabla\times\vec{J}$$

Once you've done that, let's expand on things a bit. Maxwell's equations in differential form,

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{E} = \frac{-\partial\vec{B}}{\partial t} \qquad \nabla\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

sometimes cause people to conclude odd things because they treat changing electric and magnetic fields as sources on an equal footing with charges and currents. But of course, those aren't *really* equal. Ultimately the changing electric and magnetic fields themselves came from charges and currents. Indeed, eventually all E and B fields come from charges and currents, period. The equations that this problem had you derive make that clearer: They're differential equations for E & B that explicitly lay out charges and currents as the ultimate sources of the fields.

If you ever take graduate E&M, you may have the good fortune to solve these differential equations using Green's functions to get the following integral solutions:

$$\vec{E}(x,t) = \frac{1}{4\pi\varepsilon_0}\int\left\{\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\rho(\vec{x}',t') + \frac{\vec{x}-\vec{x}'}{c|\vec{x}-\vec{x}'|^2}\frac{\partial\rho(\vec{x}',t')}{\partial t'} - \frac{1}{c^2|\vec{x}-\vec{x}'|}\frac{\partial\vec{J}(\vec{x}',t')}{\partial t'}\right\}d^3x'$$

$$\vec{B}(x,t) = \frac{\mu_0}{4\pi}\int\left\{\vec{J}(\vec{x}',t')\times\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} + \frac{\partial\vec{J}(\vec{x}',t')}{\partial t'}\times\frac{\vec{x}-\vec{x}'}{c|\vec{x}-\vec{x}'|^2}\right\}d^3x'$$

with everything in the integrand evaluated at the retarded time, as usual. These are known as Jefimenko's equations, and are the complete integral prescription for figuring out fields, given complete information about the charges and currents in the neighborhood. Hopefully you're still reading, because this is the place where I tell you that part of the credit for this problem comes from you commenting on how awesome this Jefimenko perspective is. Also examine the structure of the Jefimenko equations and comment on whether they make any intuitive sense and why (or why not, if that's how you feel).

2) (based on Pollack and Stump 12.1)

Derive the form of the Lorentz transformation by assuming the transformation is linear and does not change the perpendicular components. That is, assume a transformation of the form

$$x' = A_1(x - vt) \quad y' = y \quad z' = z \quad t' = A_2 t + A_3 x$$

Determine the constants $A_1, A_2,$ and A_3 by requiring that a flash of light produces an outgoing spherical wave of speed *c* in either the primed or the unprimed frames.

This one is neat because it shows that you can get from the postulate about the constancy of the speed of light right to the Lorentz transforms (with some very reasonable assumptions along the way).

A little hint: It's basically saying that in different frames, the distance traveled by light in every direction is *c* times the time elapsed. That means you can write an expression involving *t* and *c* about the distance to some point on an arbitrary sphere.

3) (based on Pollack and Stump 12.8)

Sketch a spacetime diagram - a diagram of the plane that has *x* on one axis and *ct* on the other. Show the following features:

a) The light cone – the set of all points that will be touched by a light ray that's emitted from coordinates $x = 0, t = 0$.

b) The trajectory of an observer at rest at the origin.

c) The trajectory of an observer who travels from $x = 0$ to $x = L$ and back, starting at $t = 0$ and moving at speed $v < c$.

Also indicate which parts of the diagram would be inaccessible to an observer starting at the origin.

4) (based on Pollack and Stump 12.16)

a) Given that J^{μ} is a four vector, write out the transformation rules for ρ and the three components of *J*.

b) Suppose that in some frame F', ρ' is zero and J' is nonzero. What are ρ and *J* in some other frame *F*? Show explicitly that $J = \rho v$ in this new frame, and explain physically why ρ is greater than ρ'.