Lagrangian formulation:

Hamilton's principle.

trajectory of a particle btw time t_1, t_2 is:

$$\delta \int_{t_1}^{t_2} (T-V)\, dt = 0$$

$T \to KE$

$V \to PE$

δ variation in path.

$\mathcal{L} \equiv T - V = f(x, \dot{x})$

if forces are conservative, $H = T + V = \text{const.}$

if $\delta \int_{t_1}^{t_2} \mathcal{L}\, dt = 0$ then integral = const.

suppose we are minimizing w.r.t a parameter α

$$\to \frac{\partial}{\partial \alpha} \int_{t_1}^{t_2} \mathcal{L}(x,\dot{x})\, dt = \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial x} \frac{\partial x}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial \alpha} \right) dt$$

our extremum path is $x(t)$

and our variation is $x(\alpha, t) = x(0,t) + \alpha\, \eta(t)$

s.t. when $\alpha = 0$ we have the optimum path.

$$\therefore \frac{\partial x}{\partial \alpha} = \eta(t) \qquad \frac{\partial \dot{x}}{\partial \alpha} = \frac{\partial \eta}{\partial t}$$

and $\eta(t_1) = \eta(t_2)$

$$\therefore \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial x} \eta(t) + \frac{\partial \mathcal{L}}{\partial \dot{x}} \frac{\partial \eta}{\partial t} \right) dt$$

$u = \frac{\partial \mathcal{L}}{\partial \dot{x}}$ $du = \frac{d}{dt}$

$dv = \frac{\partial \eta}{\partial t}$ $v = \eta$

integrate second term by parts:

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial \dot{x}} \frac{\partial \eta}{\partial t}\, dt = \cancel{\frac{\partial \mathcal{L}}{\partial \dot{x}} \eta(t) \Big|_{t_1}^{t_2}} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \eta(t)\, dt$$

0

together:

$$\int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\eta(t)\,dt = \text{const.}$$

since $\eta(t)$ is an arbitrary function, const $\to 0$
then

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$ Lagrangian eq. of motion.

[redacted] $\sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

$dz\sqrt{1 + \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2}$

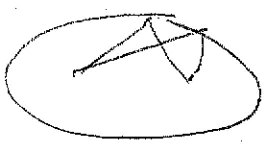
Fermat's principle

$$\delta\int n\,ds = \delta\int \underbrace{n(\vec{r})\sqrt{1 + \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2}}_{\text{optical } \mathcal{L} \,=\, n\sqrt{1+(x')^2+(y')^2}}\,dz = 0$$

so z has same role as t in classical mechanics.
note $n\,ds$ is the optical path.

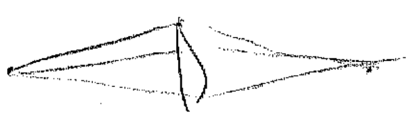
examples:

ellipse



all paths from one focus to next are the same.

lens



Field Lagrangian   HW 4.9

$$\mathcal{L} = T - U$$

for charge in static field (electric)

$$\left.\begin{array}{l} T = \frac{1}{2}mu^2 \\ U = q\phi \end{array}\right\} \quad \mathcal{L} = \frac{1}{2}mu^2 - q\phi$$

what if magnetic field is present?

anzatz:

$$\rightarrow \quad \mathcal{L} = \frac{1}{2}mu^2 + \frac{q}{c}\vec{u}\cdot\vec{A} - q\Phi$$

This works if we get the same eqns of motion. (i.e. Lorentz force eqn)

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u_i} = \frac{\partial \mathcal{L}}{\partial x_i}$$

e.g. $\quad \mathcal{L} = \frac{1}{2}mu^2 - mgx$

$$\frac{\partial \mathcal{L}}{\partial u} = \underset{\text{(momentum)}}{mu} \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u} = m\frac{du}{dt} = ma$$

$$\frac{\partial \mathcal{L}}{\partial x} = -mg = \text{Force}$$

in our case,

$$\frac{\partial \mathcal{L}}{\partial u_i} = mu_i + \frac{q}{c}A_i$$

as a vector

$$\sum \hat{x}_i \frac{\partial \mathcal{L}}{\partial u_i} = \underbrace{\vec{p} + \frac{q}{c}\vec{A}}_{\text{generalized momentum}}$$

Now evaluate $\frac{\partial \mathcal{L}}{\partial x_i} = \frac{q}{c}\frac{\partial}{\partial x_i}(\vec{u}\cdot\vec{A}) - q\frac{\partial \Phi}{\partial x_i}$

write in vector form:

$$\sum \hat{x}_i \frac{\partial \mathcal{L}}{\partial x_i} = \nabla \mathcal{L} = \frac{q}{c}\nabla(\vec{u}\cdot\vec{A}) - q\nabla\Phi$$

use a general vector ID

$$\nabla(\vec{u}\cdot\vec{A}) = (\vec{u}\cdot\vec{\nabla})\vec{A} + (\vec{A}\cdot\vec{\nabla})\vec{u} + \vec{A}\times(\nabla\times\vec{u}) + \vec{u}\times(\nabla\times\vec{A})$$

here, x, u are indep. variables. $\therefore$ terms 2,3 $\to 0$

Note $\nabla\times\vec{A} = \vec{B}$ (term 4)

For term 1:

$$\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + \sum \frac{\partial\vec{A}}{\partial x_i}\frac{\partial x_i}{\partial t} = \frac{\partial\vec{A}}{\partial t} + (\vec{u}\cdot\nabla)\vec{A}$$

$\equiv$ "total derivative"

$$\therefore (\vec{u}\cdot\nabla)\vec{A} = \frac{d\vec{A}}{dt} - \frac{\partial\vec{A}}{\partial t}$$

together:

$$\nabla\mathcal{L} = \frac{q}{c}\left(\frac{d\vec{A}}{dt} - \frac{\partial\vec{A}}{\partial t}\right) + \frac{q}{c}\vec{u}\times\vec{B} - q\nabla\Phi$$

eqn of motion is now

$$\sum_i \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial u_i} = \frac{d}{dt}\left(\vec{p} + \cancel{\frac{q}{c}\vec{A}}\right) = \nabla\mathcal{L} = \frac{q}{c}\frac{d\vec{A}}{dt} + q\overbrace{\left(-\nabla\phi - \frac{1}{c}\frac{\partial A}{\partial t}\right)}^{\vec{E}} + \frac{q}{c}\vec{u}\times\vec{B}$$

dA/dt terms cancel, so

$$\vec{F} = q\vec{E} + \frac{q}{c}\vec{u}\times\vec{B} \quad ✓$$

use $\vec{p} \to \vec{p} + \frac{q}{c}\vec{A}$ for generalized momentum

and $\mathcal{L} = \frac{1}{2}mu^2 + \frac{q}{c}\vec{u}\cdot\vec{A} - q\Phi$

Extension of Lagrangian to relativistic case.

with no forces ($\vec{E}, \vec{B} \to 0$)

$$\mathcal{L} = -m_0 c^2 \sqrt{1-\beta^2} = -\frac{m_0 c^2}{\gamma}$$

check:

1^{st} check non-rel limit - $\beta << 1$

$$\mathcal{L} = -m_0 c^2 (1 - \tfrac{1}{2}\beta^2) = \tfrac{1}{2} m_0 u^2 - \underline{m_0 c^2}$$

rest mass energy.

2^{nd} check force eqn:

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0$$

$$\frac{\partial \mathcal{L}}{\partial u_i} = -m_0 c^2 (-2u_i/c^2)(\tfrac{1}{2})(1 - u^2/c^2)^{1/2}$$
$$= m_0 u_i \gamma$$

$$\frac{d}{dt}(m_0 u_i \gamma) = 0$$ same eqn as relativistic force. eqn. 14.39.

relativistic Lagrangian is

$$\mathcal{L} = -\frac{m_0 c^2}{\gamma} + \frac{q}{c} \vec{u} \cdot \vec{A} - q\Phi$$

in 4-vector form, $\mathbb{A} = (\vec{A}, i\Phi)$

$\mathbb{U} = (\gamma \vec{u}, i\gamma c)$

$$\mathcal{L} = \frac{1}{\gamma}\left(-m_0 c^2 + \frac{q}{c}(\mathbb{U} \cdot \mathbb{A})\right)$$