Problem 1.56

Start at the origin.

(1) $\theta = \frac{\pi}{2}$, $\phi = 0$; $r : 0 \to 1$. $\mathbf{v}\cdot d\mathbf{l} = \left(r\cos^2\theta\right)(dr) = 0$. $\int \mathbf{v}\cdot d\mathbf{l} = 0$.

(2) $r = 1$, $\theta = \frac{\pi}{2}$; $\phi : 0 \to \pi/2$. $\mathbf{v}\cdot d\mathbf{l} = (3r)(r\sin\theta\, d\phi) = 3\,d\phi$. $\int \mathbf{v}\cdot d\mathbf{l} = 3\int_0^{\pi/2} d\phi = \frac{3\pi}{2}$.

(3) $\phi = \frac{\pi}{2}$; $r\sin\theta = y = 1$, so $r = \frac{1}{\sin\theta}$, $dr = \frac{-1}{\sin^2\theta}\cos\theta\, d\theta$, $\theta : \frac{\pi}{2} \to \frac{\pi}{4}$.

$$\begin{aligned} \mathbf{v}\cdot d\mathbf{l} &= \left(r\cos^2\theta\right)(dr) - (r\cos\theta\sin\theta)(r\,d\theta) = \frac{\cos^2\theta}{\sin\theta}\left(-\frac{\cos\theta}{\sin^2\theta}\right) d\ theta - \frac{\cos\theta\sin\theta}{\sin^2\theta}\,d\theta \\ &= -\left(\frac{\cos^3\theta}{\sin^3\theta} + \frac{\cos\theta}{\sin\theta}\right) d\theta = -\frac{\cos\theta}{\sin\theta}\left(\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta}\right) d\theta = -\frac{\cos\theta}{\sin^3\theta}\,d\theta. \end{aligned}$$

Therefore

$$\int \mathbf{v}\cdot d\mathbf{l} = -\int_{\pi/2}^{\pi/4} \frac{\cos\theta}{\sin^3\theta}\,d\theta = \left.\frac{1}{2\sin^2\theta}\right|_{\pi/2}^{\pi/4} = \frac{1}{2\cdot(1/2)} - \frac{1}{2\cdot(1)} = 1 - \frac{1}{2} = \frac{1}{2}.$$

(4) $\theta = \frac{\pi}{4}$, $\phi = \frac{\pi}{2}$; $r : \sqrt{2} \to 0$. $\mathbf{v}\cdot d\mathbf{l} = \left(r\cos^2\theta\right)(dr) = \frac{1}{2}r\,dr$.

$$\int \mathbf{v}\cdot d\mathbf{l} = \frac{1}{2}\int_{\sqrt{2}}^{0} r\,dr = \left.\frac{1}{2}\frac{r^2}{2}\right|_{\sqrt{2}}^{0} = -\frac{1}{4}\cdot 2 = -\frac{1}{2}.$$

Total:

$$\oint \mathbf{v}\cdot d\mathbf{l} = 0 + \frac{3\pi}{2} + \frac{1}{2} - \frac{1}{2} = \boxed{\frac{3\pi}{2}}.$$

Stokes' theorem says this should equal $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a}$

$$\begin{aligned} \nabla\times\mathbf{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, 3r) - \frac{\partial}{\partial\phi}(-r\sin\theta\cos\theta)\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(r\cos^2\theta\right) - \frac{\partial}{\partial r}(r3r)\right]\hat{\boldsymbol{\theta}} \\ &\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(-rr\cos\theta\sin\theta) - \frac{\partial}{\partial\theta}\left(r\cos^2\theta\right)\right]\hat{\boldsymbol{\phi}} \\ &= \frac{1}{r\sin\theta}[3r\cos\theta]\,\hat{\mathbf{r}} + \frac{1}{r}[-6r]\,\hat{\boldsymbol{\theta}} + \frac{1}{r}[-2r\cos\theta\sin\theta + 2r\cos\theta\sin\theta]\,\hat{\boldsymbol{\phi}} \\ &= 3\cot\theta\,\hat{\mathbf{r}} - 6\,\hat{\boldsymbol{\theta}}. \end{aligned}$$

(1) *Back face:* $d\mathbf{a} = -r\,dr\,d\theta\,\hat{\boldsymbol{\phi}}$; $(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0$. $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0$.

(2) *Bottom:* $d\mathbf{a} = -r\sin\theta\,dr\,d\phi\,\hat{\boldsymbol{\theta}}$; $(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 6r\sin\theta\,dr\,d\phi$. $\theta = \frac{\pi}{2}$, so $(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 6r\,dr\,d\phi$

$$\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = \int_0^1 6r\,dr\int_0^{\pi/2} d\phi = 6\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{2}.\quad\checkmark$$

Problem 5.10

(a) The forces on the two sides cancel. At the bottom, $B = \frac{\mu_0 I}{2\pi s} \Rightarrow F = \left(\frac{\mu_0 I}{2\pi s}\right) Ia = \frac{\mu_0 I^2 a}{2\pi s}$ (up). At the top, $B = \frac{\mu_0 I}{2\pi(s+a)} \Rightarrow F = \frac{\mu_0 I^2 a}{2\pi(s+a)}$ (down). The net force is $\boxed{\frac{\mu_0 I^2 a^2}{2\pi s(s+a)}}$ (up).

(b) The force on the bottom is the same as before, $\mu_0 I^2/2\pi$ (up). On the left side, $\mathbf{B} = \frac{\mu_0 I}{2\pi y}\hat{\mathbf{z}}$; $d\mathbf{F} = I(d\mathbf{l}\times\mathbf{B}) = I(dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}})\times\left(\frac{\mu_0 I}{2\pi y}\hat{\mathbf{z}}\right) = \frac{\mu_0 I^2}{2\pi y}(-dx\,\hat{\mathbf{y}} + dy\,\hat{\mathbf{x}})$. But the x component cancels the corresponding term from the right side, and $F_y = -\frac{\mu_0 I^2}{2\pi}\int_{s/\sqrt{3}}^{(s/\sqrt{3}+a/2)} \frac{1}{y}\,dx$. Here $y = \sqrt{3}x$, so $F_y = -\frac{\mu_0 I^2}{2\sqrt{3}\pi}\ln\left(\frac{s/\sqrt{3}+a/2}{s/\sqrt{3}}\right) = -\frac{\mu_0 I^2}{2\sqrt{3}\pi}\ln\left(1 + \frac{\sqrt{3}a}{2s}\right)$. The force on the right side is the same, so the net force on the triangle is $\boxed{\frac{\mu_0 I^2}{2\pi}\left[1 - \frac{2}{\sqrt{3}}\ln\left(1 + \frac{\sqrt{3}a}{2s}\right)\right]}$.