

11_28_07

Note Title

11/15/2006

$$\nabla^2 \psi(r, \theta, \phi) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right)$$

$$+ \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right)$$

$$+ \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} = 0$$

Laplace's equation in
spherical coordinates

Make our normal
sep. of variables guess

$$\psi(r, \theta, \phi) = R(r) P(\theta) Q(\phi)$$

So, e.g. $\frac{\partial \psi}{\partial r} = R'(r) P(\theta) Q(\phi)$

$$\nabla^2 \psi = \frac{1}{r^2} \rho \frac{d}{dr} \left(r^2 \frac{d\rho}{dr} \right)$$

$$+ \frac{\rho \phi}{r^2 \sin^2 \theta} \frac{d}{d\theta} \left(\sin^2 \theta \frac{d\phi}{d\theta} \right)$$

$$\frac{\rho \phi}{r^2 \sin^2 \theta} \frac{d^2 \phi}{d\phi^2} = 0$$

Always now divide by $\rho \phi$

$$\nabla^2 \psi = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\rho}{dr} \right)$$

$$+ \frac{1}{r^2 \sin^2 \theta} \frac{d}{d\theta} \left(\sin^2 \theta \frac{d\phi}{d\theta} \right)$$

$$+ \frac{1}{r^2 \sin^2 \theta} \frac{d^2 \phi}{d\phi^2} = 0$$

we can isolate the $Q(\varphi)$ part if we multiply by $r^2 \sin^2 \theta \Rightarrow$

$$\nabla^2 \psi = \frac{\sin^2 \theta}{r} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right)$$

$$+ \frac{\sin \theta}{\rho} \frac{d}{d\theta} \left(\sin \theta \frac{dP}{d\theta} \right)$$

$$= - \frac{1}{Q} \frac{d^2 Q}{d\varphi^2} = m^2$$

$$Q'' + m^2 Q = 0$$

first ODE

$$\text{so } Q(\varphi) \propto e^{im\varphi}$$

Does m have to be an integer?

Suppose $m = .1$

$$Q(0) = 1$$

$$Q(2\pi) = e^{i \cdot .1 \cdot 2\pi} \neq 1$$

Like a Boundary condition
the periodicity forces
 m to be an integer

Back to r, θ :

$$\frac{\sin^2 \theta}{r} \frac{d}{dr} \left(r^2 \frac{dr}{dr} \right) + \frac{\sin \theta}{p} \frac{d}{d\theta} \left(\sin \theta \frac{dp}{d\theta} \right) = m^2$$

Divide by $\sin^2 \theta$ and move
 θ part to RHS

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right)$$

$$= -\frac{1}{\sin^2 \theta} \frac{1}{p} \frac{d}{d\theta} \left(\sin^2 \theta \frac{dp}{d\theta} \right) + \frac{m^2}{\sin^2 \theta}$$

r on the left θ on right

r is easiest to do first

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = k^2$$

Guess a trial solution

$$R(r) = A r^\alpha$$

$$R' = A \alpha r^{\alpha-1}$$

$$r^2 R' = A \alpha r^{\alpha+1}$$

$$\frac{1}{r} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) =$$

$$\frac{1}{A r^\alpha} \cdot A (\alpha+1) \alpha r^\alpha = k^2$$

$$\Rightarrow \alpha(\alpha+1) = k^2$$

So far we don't know about k^2 but

suppose $k^2 = \ell(\ell+1)$

$$\alpha(\alpha+1) = \ell(\ell+1)$$

$$\Leftrightarrow (\alpha - \ell)(\alpha + (\ell+1)) = 0$$

So either

$$\boxed{\begin{array}{l} \alpha = \ell \quad \text{or} \\ \alpha = -(\ell+1) \end{array}}$$

recap: we can get solutions to the radial part of Laplace's eqn if we write the separation constant as

$$k^2 = \ell(\ell+1)$$

Then $R(r) = A r^\alpha$ works for $\alpha = \ell$, $\alpha = -(\ell+1)$

So

$$R(r) = A_\ell r^\ell + B_\ell r^{-(\ell+1)}$$

second ODE solution

Finally we get to the θ part.

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dp}{d\theta} \right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2 \theta} \right] p = 0$$

Legendres equation

Exercise Let $x = \cos \theta$

Show that $\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dp}{d\theta} \right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2 \theta} \right] p = 0$ becomes

$$\frac{d}{dx} \left[(1-x^2) \frac{dp}{dx} \right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2} \right] p(x) = 0$$

Solutions of this equation involve ℓ and m indices ℓ, m .

Lets call them

$$P_{\ell, m}(x)$$

$$\text{Then } \psi(r, \theta, \phi) = R(r) P(\theta) \phi(\phi)$$

$$= \begin{cases} r^{\ell} P_{\ell m}(\cos\theta) e^{im\phi} \\ r^{-(\ell+1)} P_{\ell m}(\cos\theta) e^{im\phi} \end{cases}$$

$$\underbrace{P_{\ell m}(\cos\theta) e^{im\phi}} \equiv \underbrace{Y_{\ell m}(\theta, \phi)}$$

Legendre
polynomials

spherical
harmonics

Major Result!

Any solution of $\nabla^2 \phi = 0$
can be written as

$$\phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l (A_{lm} r^l + B_{lm} r^{-(l+1)}) Y_{lm}$$

we have not yet proved
that m runs from
 $-l, l$ but enough for
today.

$$Y_{00} = \sqrt{\frac{1}{4\pi}}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{1\pm 1} = \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

⋮

See my lecture notes on
"sep of variables & special functions"
wiki week of 11/26.

Also BOAS ch. 13 sec. 7