ch. 3, Sec 11 Boas 10/2/06

Note Title 10/2/2006

recall

$$\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

has 2 Eigenvalue/Eigenvector pairs

$6, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $4, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_1 \quad \vec{q}_1 \qquad\qquad \lambda_2 \quad \vec{q}_2$

notice that $(\vec{q}_1, \vec{q}_2) = 0$

These vectors are orthogonal.

Let's normalize them

$\vec{q}_1 \rightarrow \frac{1}{\sqrt{2}}(1, 1)$

$\vec{q}_2 \rightarrow \frac{1}{\sqrt{2}}(1, -1)$

So $(\vec{q}_1, \vec{q}_2) = (\vec{q}_2, \vec{q}_1) = 0$

$$(\vec{q}_1, \vec{q}_1) = (\vec{q}_2, \vec{q}_2) = 1$$

Together, these eigenvectors form an orthogonal matrix

$$Q \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$Q^T Q = Q Q^T = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Similarly, the eigenvalues form a diagonal matrix

$$\Lambda \equiv \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}$$

Lambda

Amazing result

$$Q \Lambda Q^T = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$$

since $Q^T Q = I$

$$\boxed{\Lambda = Q^T \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} Q}$$

This is called the diagonalization of the original matrix.

Diagonalization of matrices can be thought of as a coordinat transformation in which linear systems becom simple

NB today we treat only symmetric matrices

Many (most) matrices that arise in physics are symm.

Nonsymmetric problems are much harder. Later consider $A\vec{x}=\vec{y}$ ☆

suppose $Q$ is an orthogonal matrix, then

$$Q^TQ = \underline{I} \Rightarrow$$

$$A\,Q^T Q\,\vec{x} = \vec{y} \qquad \text{insert } I.$$

$$Q\,A\,Q^T\,\underbrace{Q\,\vec{x}}_{\vec{x}^r} = \underbrace{Q\,\vec{y}}_{\vec{y}^r} \qquad \text{multiply by } Q$$

$$Q\,A\,Q^T\;\vec{x}^r = \vec{y}^r$$

if $Q\,A\,Q^T = \Lambda$ then

☆ $\Rightarrow$ $\Lambda\,\vec{x}^r = \vec{y}^r$

$$= \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}_{\vec{x}^r} = \underbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}_{\vec{y}^r}$$

immediately we have

$$x_1^r = \frac{1}{\lambda_1} y_1^r$$

$$\vdots$$

$$x_i^r = \frac{1}{\lambda_i} y_i^r$$

The original solution $\vec{x}$ is related to $\vec{x}^r$ by

$$Q\vec{x} = \vec{x}^r$$

$$\Rightarrow \quad \vec{x} = Q^T \vec{x}^r = Q^T \Lambda^{-1} \vec{y}^r$$

$$\boxed{\vec{x} = Q^T \Lambda^{-1} Q \, \vec{y}}$$

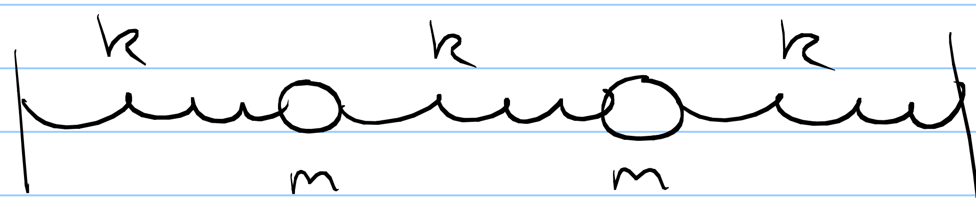
$$\vec{x} = A^{-1} y$$

Key idea: in the new coordinates the problem becomes easy

# Important example

diagonalizing coupled oscillator problem leads to uncoupled oscillators!

## Example




Can show

$$\ddot{x}_1 = -2\omega_0^2 x_1 + \omega_0^2 x_2$$

$$\ddot{x}_2 = -2\omega_0^2 x_2 + \omega_0^2 x_1$$

Seek solutions: $x_1 = Ae^{i\omega t}$

$x_2 = Be^{i\omega t}$

Plug into ☆

$$\underbrace{\begin{pmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 \end{pmatrix}}_{\text{matrix}} \underbrace{\begin{pmatrix} A \\ B \end{pmatrix}}_{\text{vect}} = \underbrace{\omega^2}_{\text{scalar}} \underbrace{\begin{pmatrix} A \\ B \end{pmatrix}}_{\text{vector}}$$

$$K \vec{u} = \omega^2 \vec{u}$$

$$\vec{u} = \begin{pmatrix} A \\ B \end{pmatrix} \qquad K = \begin{pmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 \end{pmatrix}$$

Lets solve this problem

characteristic polynomial $\equiv$

$$\mathrm{Det}(A - \lambda I)$$

$$\mathrm{Det}\begin{pmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{pmatrix}$$

$$= (2\omega_0^2 - \omega^2)^2 - \omega_0^4 = 0$$

$$\Rightarrow \quad 2\omega_0^2 - \omega^2 = \pm \omega_0^2$$

$$\Rightarrow \underline{\omega_\pm^2 = 2\omega_0^2 \mp \omega_0^2}$$

$$\boxed{\begin{aligned} \omega_+^2 &= 3\omega_0^2 \\ \omega_-^2 &= \omega_0^2 \end{aligned}} \quad \text{e-values}$$

plug ~~these~~ into

$$\underbrace{\begin{pmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 \end{pmatrix}}_{\text{matrix}} \underbrace{\begin{pmatrix} A \\ B \end{pmatrix}}_{\text{vect}} = \underbrace{\omega^2}_{\text{scalar}} \underbrace{\begin{pmatrix} A \\ B \end{pmatrix}}_{\text{vector}}$$

$$2\omega_0^2 A - \omega_0^2 B = 3\omega_0^2 A$$

$$-\omega_0^2 A + 2\omega_0^2 B = \omega_0^2 B$$

$3\omega_0^2$ $\Rightarrow$ $-\omega_0^2 B = \omega_0^2 A$

$\Rightarrow B = -A$

eigenvector $= \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\omega_0^2$ $\Rightarrow$ $-\omega_0^2 A = -\omega_0^2 B$

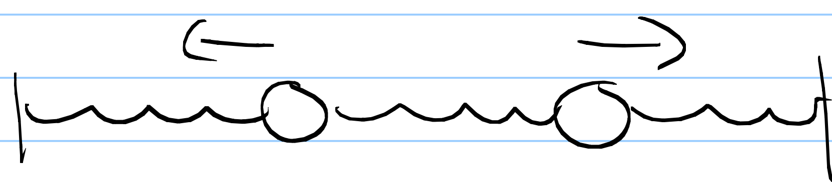
$\Rightarrow A = B$

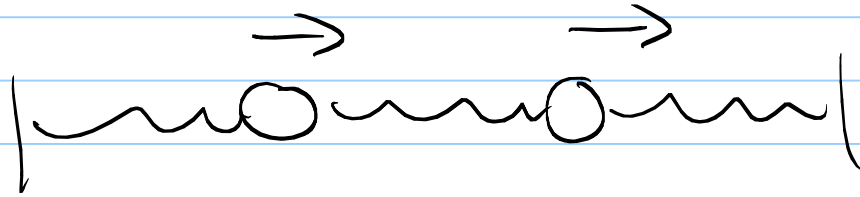
eigenvector $= \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

---

$3\omega_0^2$, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$   $\omega_0^2$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

fast motion   slow motion

$3\omega_0^2$  out of phase

$\omega_0^2$  in phase original coupled EQN'S.

$$K\vec{u} = \omega^2 \vec{u}$$

$$\vec{u} = \begin{pmatrix} A \\ B \end{pmatrix} \qquad K = \begin{pmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 \end{pmatrix}$$

$$K = Q \Lambda Q^T$$

$$\Rightarrow \quad Q\Lambda Q^T \vec{u} = \omega^2 \vec{u}$$

$$\Rightarrow \quad \Lambda \underbrace{Q^T \vec{u}}_{\vec{u}^r} = \omega^2 \underbrace{Q^T \vec{u}}_{\vec{u}^r}$$

$$\boxed{\Lambda \vec{u}^r = \omega^2 \vec{u}^r}$$

The equations are now uncoupled since $\Lambda$ is diagonal

$$\boxed{\Lambda \vec{u}^r = \omega^2 \vec{u}^r}$$

$$\begin{bmatrix} 3\omega_0^2 & 0 \\ 0 & \omega_0^2 \end{bmatrix} \begin{bmatrix} u_1^r \\ u_2^r \end{bmatrix} = \omega^2 \begin{bmatrix} u_1^r \\ u_2^r \end{bmatrix}$$

$$3\omega_0^2 u_1^r = \omega^2 u_1^r$$

$$\omega_0^2 u_2^r = \omega^2 u_2^r$$

---

Example of an important application.

suppose $A = Q \Lambda Q^T$

$$A^2 = A \cdot A = (Q \Lambda \underbrace{Q^T)(Q}_{Q^T Q = I} \Lambda Q^T)$$

$$= Q \Lambda^2 Q^T$$

$$\vdots$$

$$A^N = Q\Lambda^N Q^T$$

e.g. since $e^x = 1 + x + \frac{x^2}{2} + \cdots$

we guess that

$$e^A = I + A + A^2 + \cdots$$

$$= Q\Lambda^0 Q^T + Q\Lambda^1 Q^T + Q\Lambda^2 Q^T + \cdots$$

$$= Q\left[\Lambda^0 + \Lambda^1 + \Lambda^2 + \cdots\right] Q^T$$

In QM this arises in the time-evolution operator for Schrödinger's equation