Note Title 11/3/2006

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Recap of sampling theorem

1) For a bandlimited fcn

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-2\pi f_c}^{2\pi f_c} F(\omega)\, e^{-i\omega t}\, d\omega$$

2) Hence $F(\omega)$ can be expressed as a Fourier series and F.S. have discrete frequencies

$$F(\omega) = \sum_{N=-\infty}^{\infty} \phi_N \, e^{i\omega N/2f_s}$$

$$\phi_N = \frac{1}{\sqrt{2\pi}\; 2f_s} \int_{-2\pi f_c}^{2\pi f_c} F(\omega)\, e^{-i\omega N/2f_c} \, d\omega$$

compare this with

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-2\pi f_c}^{2\pi f_c} F(\omega) e^{-i\omega t} \, d\omega$$

evaluate t at $\frac{N}{2f_c}$

if $f(t)$ is not band limited

then its frequency content $F(\omega)$

is continuous.

For B-L functions the freq. content is discrete.

Further, the function $f(t)$ is completely specified by its discrete samples take ever $\frac{1}{2f_c} = \Delta t$ sampling period

$$\phi_N = \frac{f(N/2f_c)}{2f_c}$$

$$f(t) = \sum_{N=-\infty}^{\infty} f(N/2f_c) \frac{\sin(\pi(2f_c t - N))}{\pi(2f_c t - N)}$$

The sampling theorem

Mathematica example

implications

$\Delta t = \frac{1}{2f_c}$ —— max freq. in data

You must sample with a Δt no larger than $\frac{1}{2f_{max}}$ where f_{max} is the max. frequency in your data.

Equivalently: given a sampling period Δt

then $f_{max} = \frac{1}{2\Delta t}$

Nyquist freq.

The sampling frequency

is the number of samples per second.

Eg. a sound card might have a samp. f.

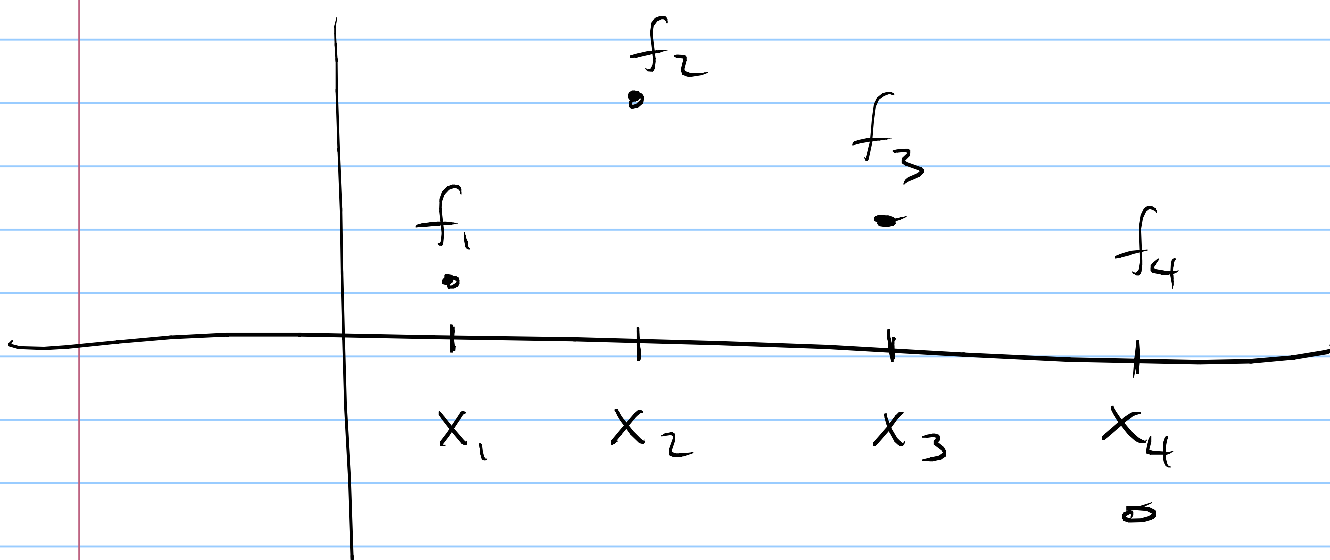
$$\boxed{\begin{aligned} f_s &= 96\,\text{KHZ} \\ \Rightarrow \Delta t &= \frac{1}{96 \times 10^3}\,\text{s} \approx 10\,\mu\text{s} \end{aligned}}$$

Hence

$$\boxed{f_{\text{Nyquist}} = \frac{1}{2\Delta t} = 48\,\text{KHZ}}$$

$$f_{\text{Nyquist}} = \frac{1}{2} f_{\text{sampling}}$$

The discrete Fourier transform.



N equally spaced data

$$x_k = \frac{2\pi k}{N} \quad \Big\} \text{ for convenience}$$

$$k = 0, 1, 2, \ldots N-1$$

(f_k, x_k)

Consider a trigonometric polynomial of order N

$$p(x) \equiv \sum_{n=0}^{N-1} c_n e^{inx}$$

we want this polynomial to interpolate the data. i.e. the curve p(x) should go through each data point exactly.

$$f_k = p(x_k) = \underbrace{\sum_{n=0}^{N-1} c_n e^{in2\pi k/N}}$$

This is actually a matrix vector mult.

$$Q \cdot \vec{c} \quad \text{where}$$

$$(\vec{c})_n = c_n$$

$$Q_{nk} = e^{i\frac{2\pi}{N} nk}$$

$$(\vec{f})_k = f_k$$

$$\vec{f} = Q \cdot \vec{c}$$

Show that $Q^{*T} Q = I$

This is the complex analog of an orthogonal matrix ✓

Q is unitary $\Longleftrightarrow$

$$Q^{\dagger} Q = Q Q^{\dagger} = I$$

where $Q^{\dagger} = Q^{*T}$ adjoint or Hermitian

dagger

Hence

$$\vec{c} = Q^{+} f$$

$$\boxed{c_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{-i 2\pi/N \, nk} f_k}$$

Discrete fourier trans.

DFT

$$P(x) = \sum_{n=0}^{N-1} c_n e^{ikx}$$

FFT

work DFT $O(N^2)$

FFT $O(N \ln N)$