10 - 29 - 07

Note Title 10/28/2007

The probability density function (PDF) of the Gaussian distribution with mean μ and standard deviation σ is

$$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

HW 8 #6

If you want to the probability that a number will be between a and b, this is

$$\frac{1}{\sqrt{2\pi}\,\sigma} \int_a^b e^{-(x-\mu)^2/2\sigma^2} dx$$

If you try this integral in Mathematica you'll get the following

$$\frac{1}{2}\left[\mathrm{Erf}\left[\frac{b-\mu}{\sigma\sqrt{2}}\right] - \mathrm{Erf}\left[\frac{a-\mu}{\sqrt{2}\,\sigma}\right]\right]$$

Erf stands for error function

$$\boxed{\mathrm{Erf}[x] = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy}$$

Example: Approximate the error fcn near $x=1$ to within 1%.

$$\frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\, dx = \frac{2}{\sqrt{\pi}} \int_0^x \left[1 - y^2 + \frac{y^4}{2} - \frac{y^6}{6} + \frac{y^8}{24} \cdots\right] dy$$

$$= \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} \cdots\right]$$

at 1 this approx. looks like

$$\frac{2}{\sqrt{\pi}} \left[1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \cdots\right]$$

So if we stop after 4 terms the error is less than 1%.

in fact $\left| \frac{2}{\sqrt{\pi}} \left[1 - 1/3 + \frac{1}{10} - \frac{1}{42}\right] - Erf[1] \right|$

$\cong 004$

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ex. Let mean $\mu = 0$, Std. dev $\sigma = 1$

$P[-1 \le x \le 1] = .683$ 68%

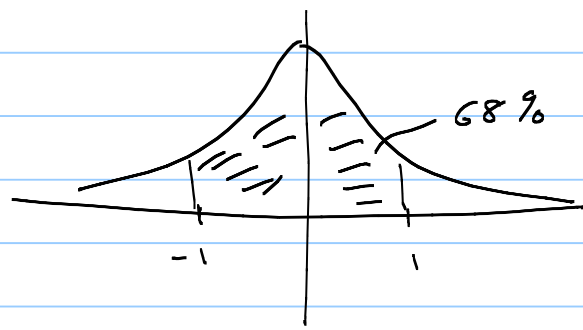
"1-sigma"

$P[-2 \le x \le 2] = .954$ 95%

"2-sigma"

$P[-3 \le x \le 3] = .997$ 99.7%

ie 99.7 % of the area under a Gaussian curve is within 3 standard deviations of the mean



Error bars are usually $\pm 1\sigma$ but this denotes only 68 % confidence

many applications of Gaussians
- Central limit theorem
- Maxwell velocity distribution
- Lasor pulses
- Quantum wave packets

No real monochromatic signals in nature (they would have to be infinitely long)

Lets start with FT of a sin

FT of $\sin \omega_0 t$ $\leftarrow$ infinitely long

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin \omega_0 t \; e^{i\omega t} \, dt$$

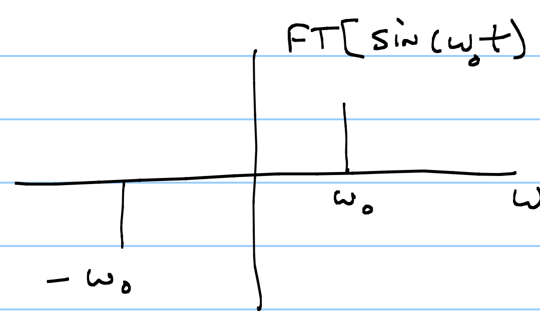
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2i} \left[e^{i\omega_0 t} - e^{-i\omega_0 t} \right] e^{i\omega t} \, dt$$

$$= \frac{1}{2i\sqrt{2\pi}} \left\{ \underbrace{\int_{-\infty}^{\infty} e^{i(\omega+\omega_0)t} \, dt} - \int_{-\infty}^{\infty} e^{i(\omega-\omega_0)t} \, dt \right.$$

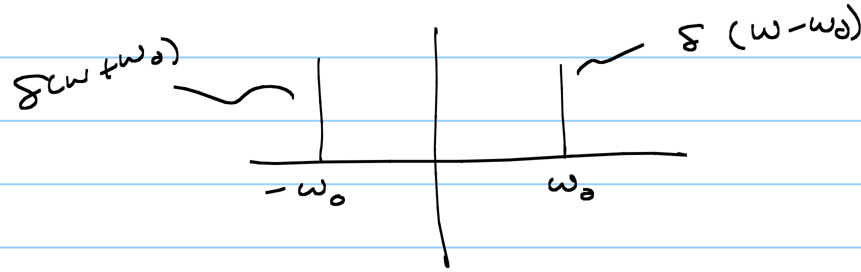
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha t} \, dt = \sqrt{2\pi} \, \delta(\alpha) .$$

$$\frac{\sqrt{2\pi}}{2i} \left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0) \right]$$

$$= i\sqrt{\frac{\pi}{2}} \left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0) \right]$$



So the power spectrum is



Digresson on FT[$\delta(t)$]

Using the basic property of the δ-function

$$\int_{-\infty}^{\infty} \delta(t-t_0)\, e^{i\omega t}\, dt$$

$$= e^{i\omega t_0}$$

so,

$$FT[\delta(t-t_0)] = \frac{1}{\sqrt{2\pi}} e^{i\omega t_0}$$

$$\Rightarrow FT^{-1}\left[\frac{1}{\sqrt{2\pi}} e^{i\omega t_0}\right] = \delta(t-t_0)$$

i.e. $$\delta(t-t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t_0}\, e^{-i\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(t_0-t)\omega}\, d\omega$$

$$\Rightarrow \boxed{\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\omega}\, d\omega}$$

This is another way of saying that

$$\sqrt{2\pi}\,\delta(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 1\, e^{i\omega t}\, d\omega$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 1\, e^{-i\omega t}\, d\omega}_{FT[1]}$$

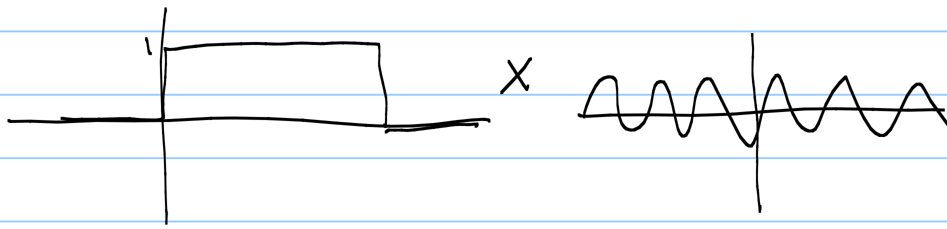
verify this by change of variable; mind the integ. limits!

$$\boxed{FT[1] = \sqrt{2\pi}\,\delta(t)}$$

$$FT[e^{i\omega_0 t}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega_0 t}\, e^{i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} e^{i(\omega_0+\omega)t}\, dt}_{\delta(\omega_0+\omega)}$$

Now what happens if we truncate the sin?



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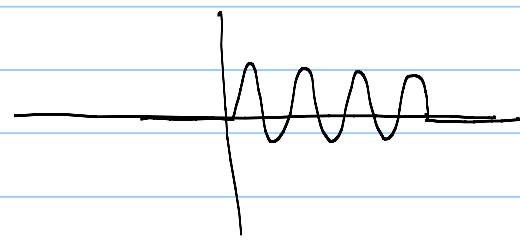
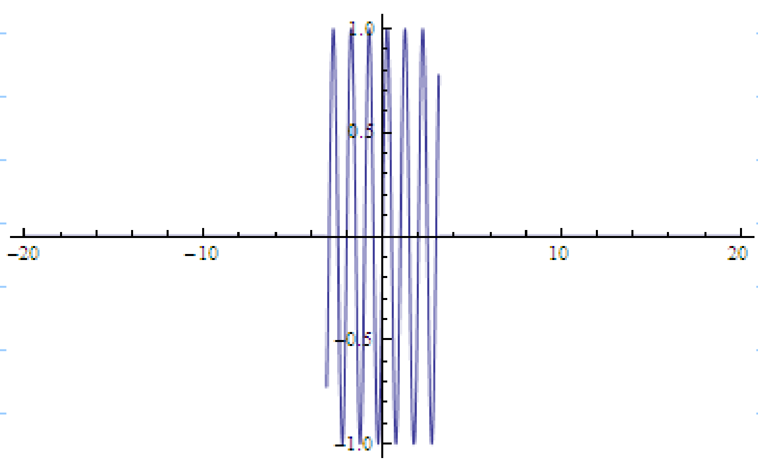
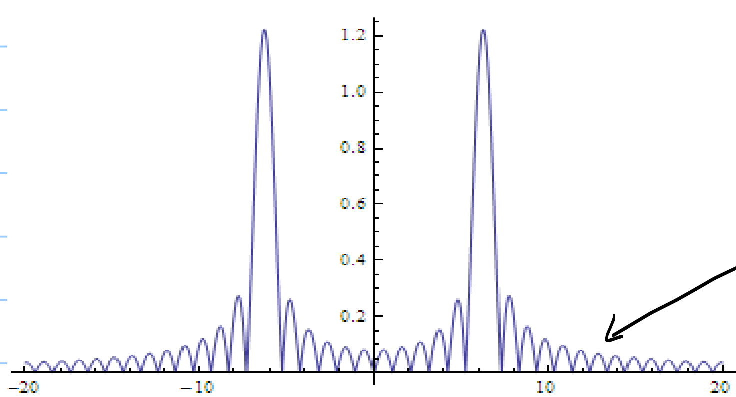


can't do this analytically, so use Mathematica

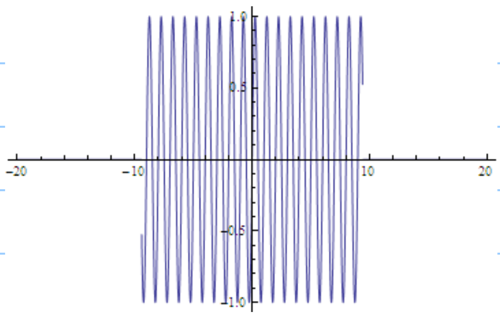


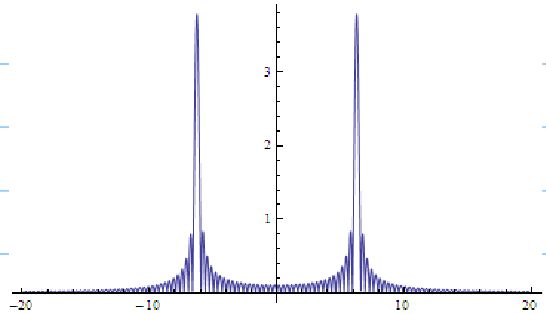
A finite piece of a sinusoid is called a tone burst



ringing due to sharp edges of Boxcar

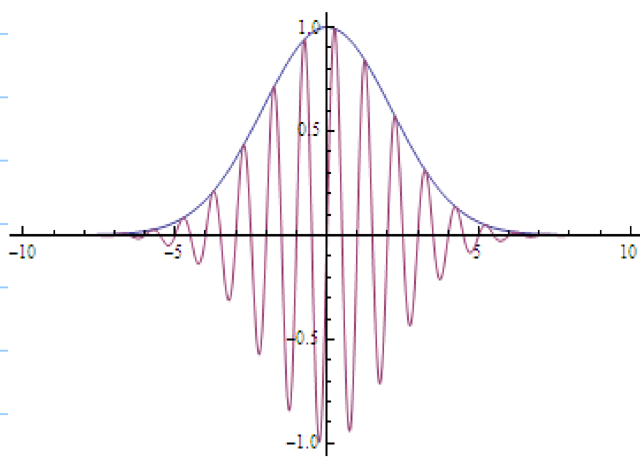
Try a smoother truncation "taper"



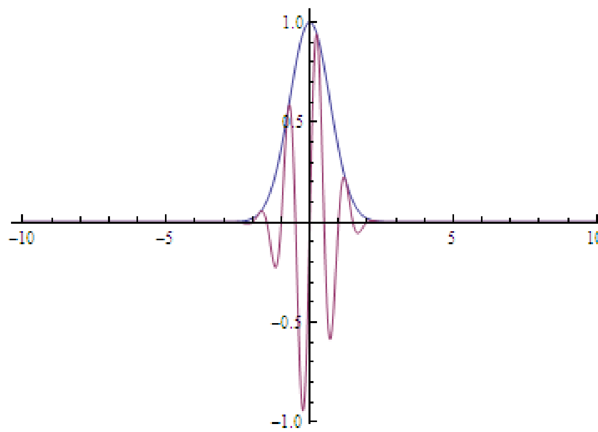


Longer toneburst $\Rightarrow \delta$

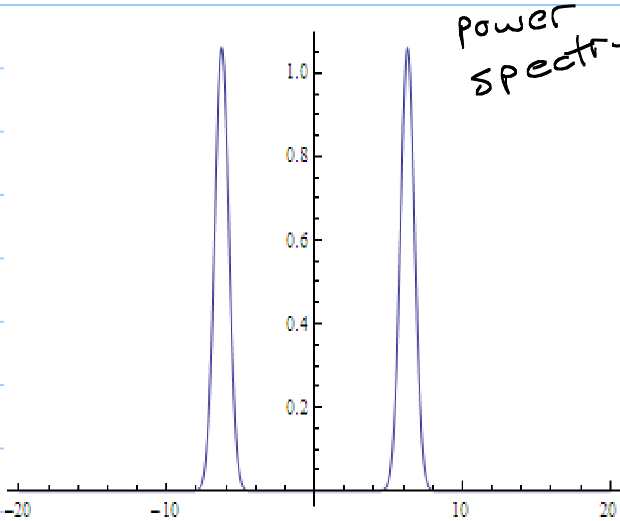
Gaussian tapers $\longleftrightarrow$ Gaussian wavepackets



Out[32]=



power spectrum



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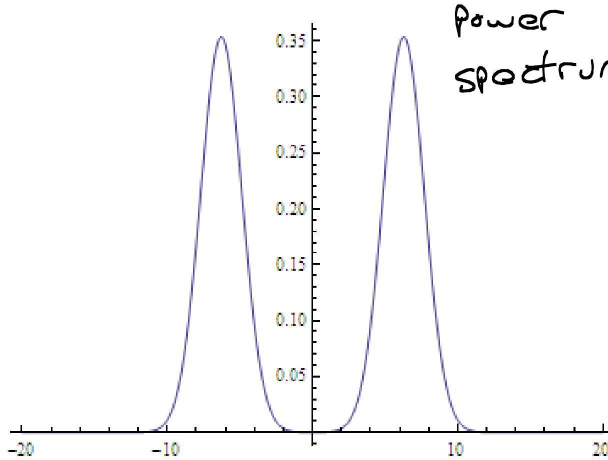
power spectrum



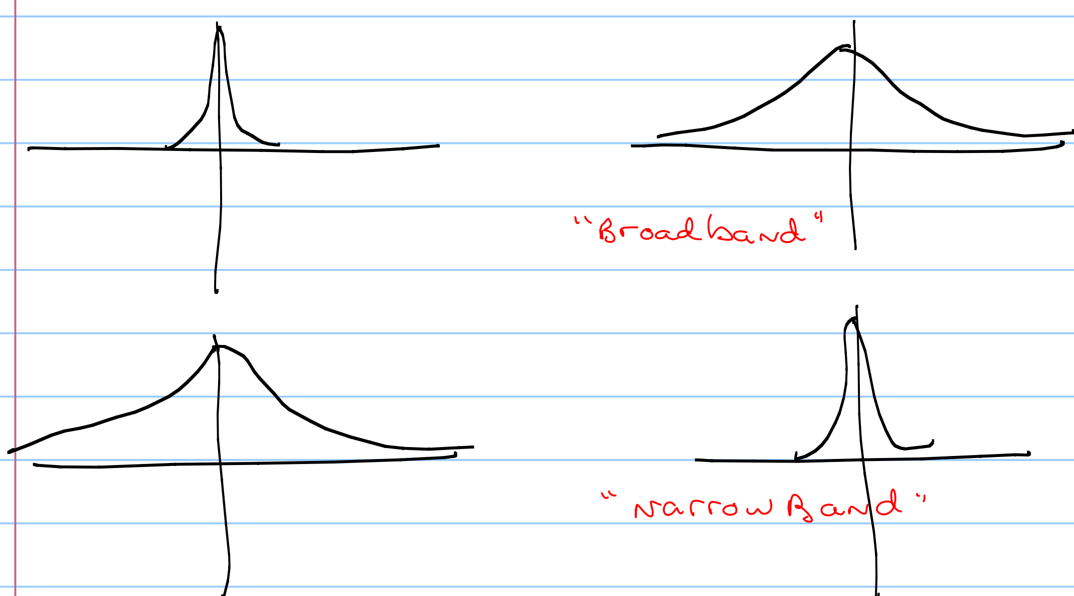
The narrower the pulse the broader the spectrum

In HW you will show:

$$FT\left[\frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(t-m)^2/2\sigma^2}\right] = \frac{1}{\sqrt{2\pi}}\, e^{im\omega}\, e^{-\omega^2\sigma^2/2}$$

If mean $m = 0$, then the FT is a Gaussian:

$$\frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-t^2/2\sigma^2} \qquad\qquad \frac{1}{\sqrt{2\pi}}\, e^{-\omega^2\sigma^2/2}$$



Eg $\quad$ 1 NS pulsed Yag laser

$\sigma = 1 \times 10^{-9}\, s$ $\quad$ 1 Nanosec

$\Downarrow$

$\frac{1}{\sigma} = 10^{9}\, Hz$ $\quad$ 1 GHz

10 fs $\quad$ Ti - Sapphire

$\sigma = 1 \times 10^{-14}\, s$

$\frac{1}{\sigma} = 1 \times 10^{14}\, Hz = 100\, THz$