Tilted window: ray propagation

- Calculate phase shift caused by the insertion of the window into an interferometer.
- Ray optics:
  - Add up optical path for each segment
  - Subtract optical path w/o window

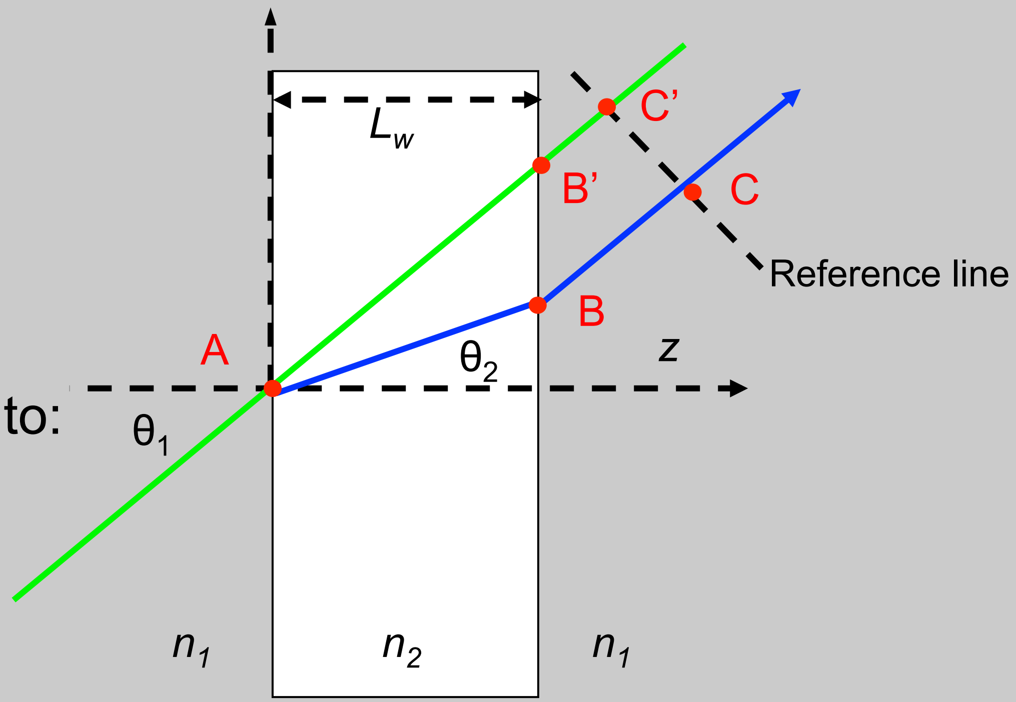
$$\Delta d = nL_{AB} + L_{BC} - L_{AB'} - L_{B'C'}$$

$$L_{AB} = \frac{L_w}{\cos\theta_2} \quad L_{AB'} = \frac{L_w}{\cos\theta_1}$$

$$L_{BC} = L_{B'C'} + L_{BB'}\sin\theta_1$$

  - Use Snell's Law to reduce to:

$$\Delta d = n\,L_w \cos\theta_2 - L_w \cos\theta_1$$



Tilted window: wave propagation

- Write expression for tilted plane wave

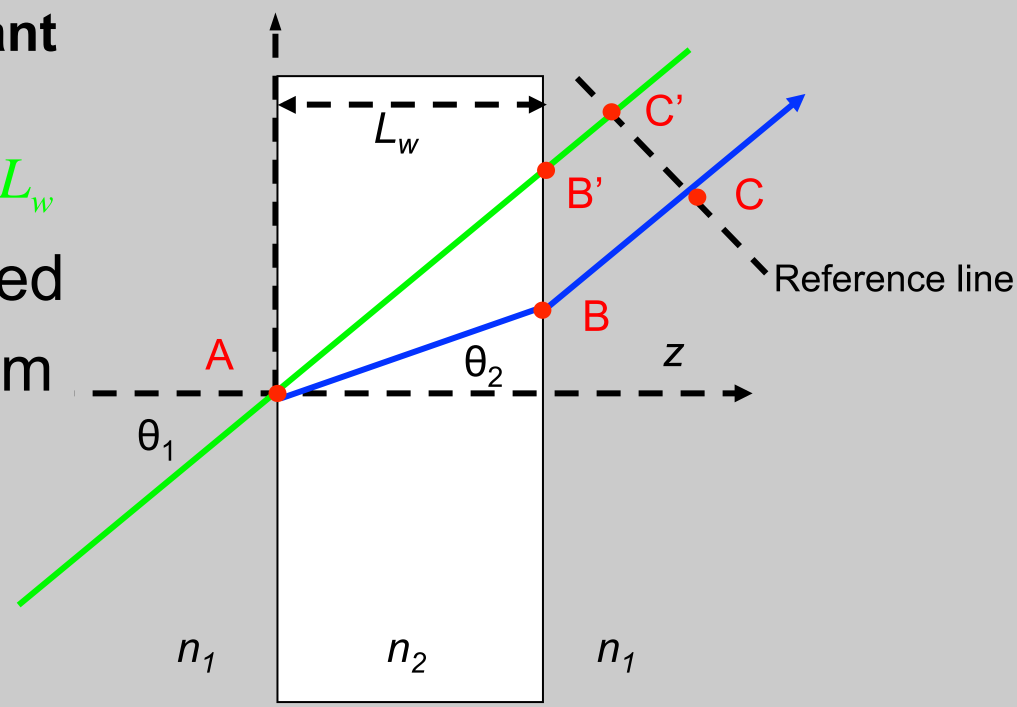
$$E(x,z) = E_0 \exp\left[i\left(k_x x + k_z z\right)\right] = E_0 \exp\left[i\frac{\omega}{c}n\left(x\sin\theta_2 + z\cos\theta_2\right)\right]$$

- Snell's Law: phase across surfaces is conserved

$$k_x x = \frac{\omega}{c} n \sin\theta$$ **is constant**

$$\Delta\phi = \left(k_2 \cos\theta_2\right) L_w - \left(k_1 \cos\theta_1\right) L_w$$

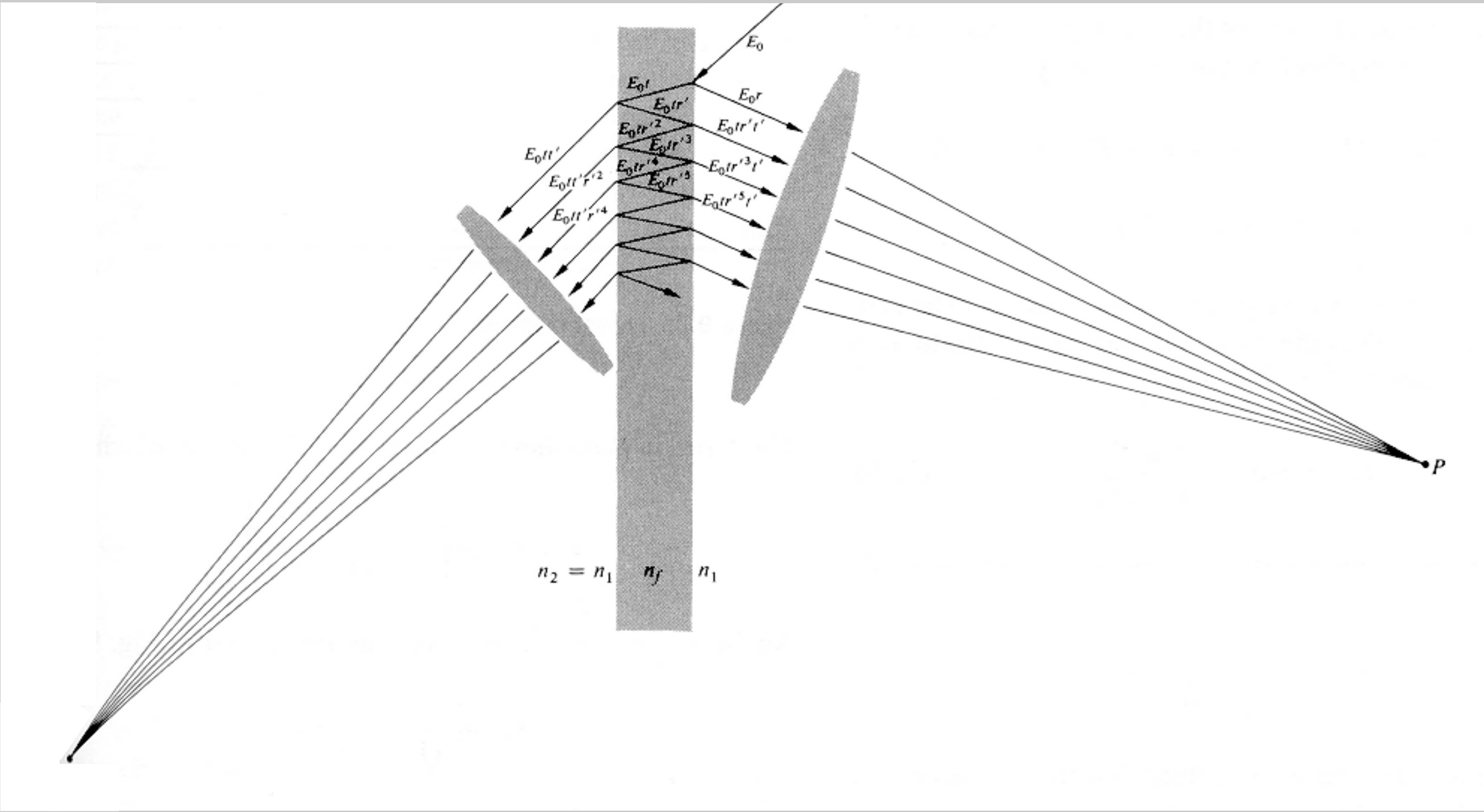
- This approach can be used to calculate phase of prism pairs and grating pairs



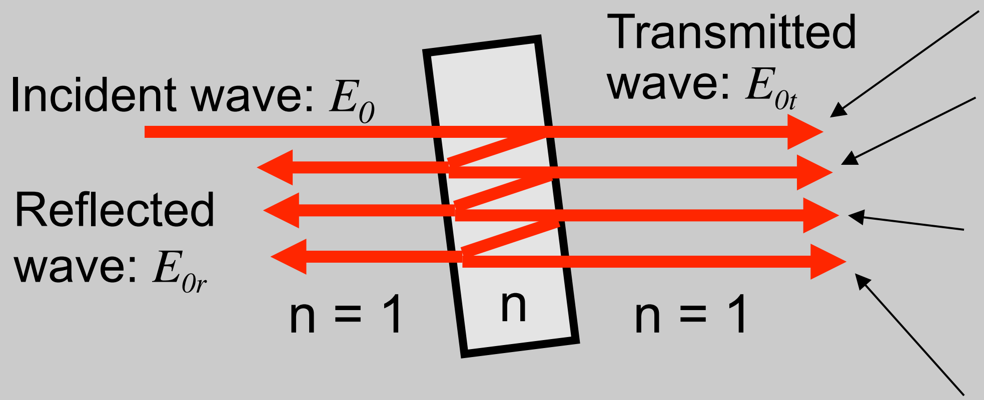
Multiple-beam interference:
The Fabry-Perot Interferometer or Etalon

A Fabry-Perot interferometer is a pair of **parallel** surfaces that reflect beams back and forth. An etalon is a type of Fabry-Perot etalon, and is a piece of glass with parallel sides.

The transmitted wave is an infinite series of multiply reflected beams.



Multiple-beam interference: general formulation



$$t\,t'\,e^{i\delta/2}\,E_0$$

$$t\,t'\,e^{i\delta/2}\left(r'\right)^2 e^{i\delta}\,E_0$$

$$t\,t'\,e^{i\delta/2}\left(\left(r'\right)^2 e^{i\delta}\right)^2 E_0$$

$$t\,t'\,e^{i\delta/2}\left(\left(r'\right)^2 e^{i\delta}\right)^3 E_0$$

r, t **= reflection, transmission coefficients from air to glass**
r', t' **=   "   "   "   from glass to air**

δ = round-trip phase delay inside medium = $k_0(2\,n\,L\cos\theta_t)$

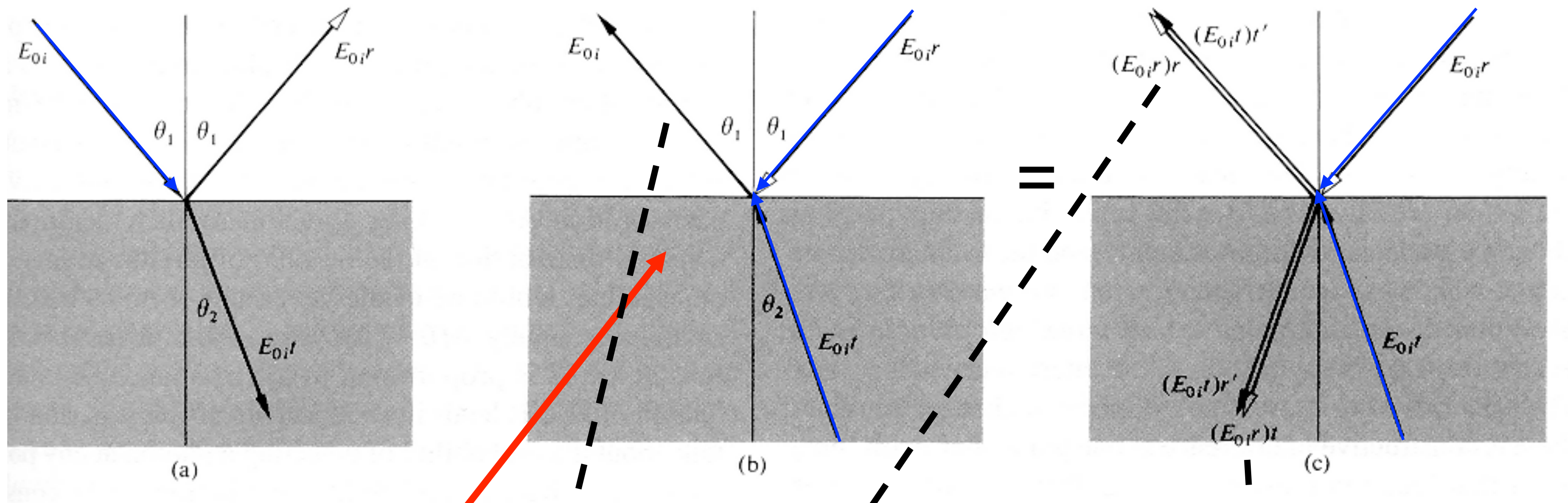
Transmitted wave:

$$E_{0t} = tt'e^{-i\delta/2}E_0\left(1+\left(r'\right)^2 e^{i\delta}+\left(\left(r'\right)^2 e^{i\delta}\right)^2+\left(\left(r'\right)^2 e^{i\delta}\right)^3+\ldots\right)$$

Reflected wave:

$$E_{0r} = rE_0 + tt'r'e^{i\delta}E_0 + tt'r'\left(\left(r'\right)^2 e^{i\delta}\right)^2 E_0 + \ldots$$

Stokes Relations for reflection and transmission



"Time reversal:"
Same amplitudes, reversed propagation direction

$$E_{oi} = (E_{oi}r)r + (E_{oi}t)t'$$
$$\therefore tt' = 1 - r^2$$

$$(E_{oi}t)r' + (E_{oi}r)t = 0$$
$$\therefore r' = -r$$

Notes:

- **relations apply to angles connected by Snell's Law**
- **true for any polarization, but not TIR**
- **convention for which interface experiences a sign change can vary**

Fabry-Perot transmission

Stokes' relations

$r' = -r$

$r'^2 = r^2$

$tt' = 1 - r^2$

The transmitted wave field is:

$$E_{0t} = tt' e^{i\delta/2} E_0 \left(1 + (r')^2 e^{i\delta} + \left((r')^2 e^{i\delta}\right)^2 + \left((r')^2 e^{i\delta}\right)^3 + \ldots\right)$$

$$= tt' e^{i\delta/2} E_0 \left(1 + r^2 e^{i\delta} + \left(r^2 e^{i\delta}\right)^2 + \left(r^2 e^{i\delta}\right)^3 + \ldots\right)$$

$$\Rightarrow \quad E_{0t} = tt' E_0 / \left(1 - r^2 e^{-i\delta}\right)$$

Where:

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

Power transmittance:

$$T \equiv \left|\frac{E_{0t}}{E_0}\right|^2 = \left|\frac{tt' e^{i\delta/2}}{1 - r^2 e^{i\delta}}\right|^2 = \frac{\left(tt'\right)^2}{(1 - r^2 e^{+i\delta})(1 - r^2 e^{-i\delta})}$$

$$= \left[\frac{\left(tt'\right)^2}{\{1 + r^4 - 2r^2 \cos(\delta)\}}\right] = \left[\frac{(1-r^2)^2}{\{1 + r^4 - 2r^2[1 - 2\sin^2(\delta/2)]\}}\right] = \left[\frac{(1-r^2)^2}{\{1 - 2r^2 + r^4 + 4r^2 \sin^2(\delta/2)]\}}\right]$$

Dividing numerator and denominator by $(1-r^2)^2$

$$T = \frac{1}{1 + F \sin^2\left(\delta/2\right)}$$

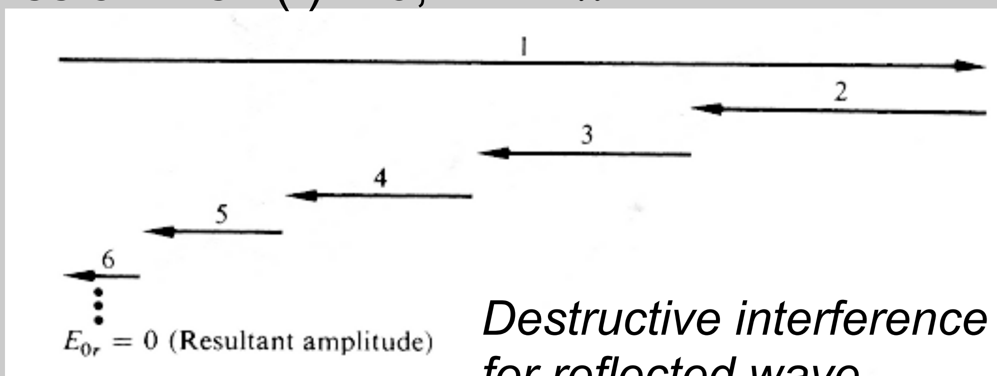
where: $F = \left[\frac{2r}{1-r^2}\right]^2$

Multiple-beam interference: simple limits

Reflected waves

$$T = \frac{1}{1 + F\sin^2(\delta/2)}$$

Full transmission: sin( ) = 0, d = 2 π m

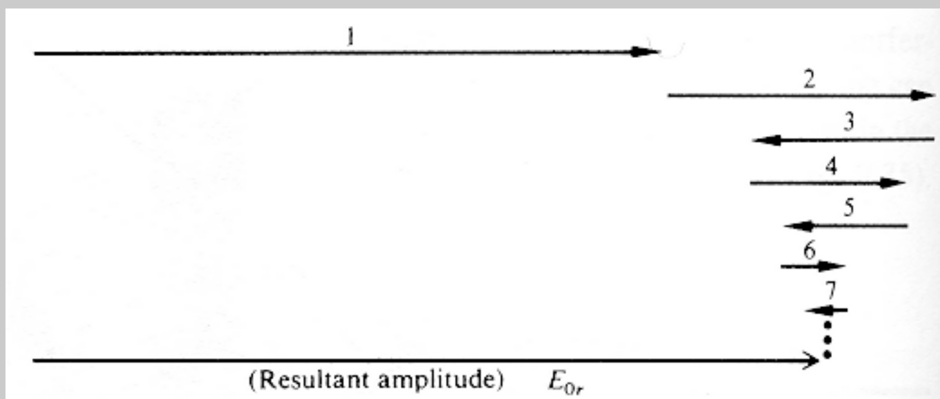


1st reflection

internal reflections

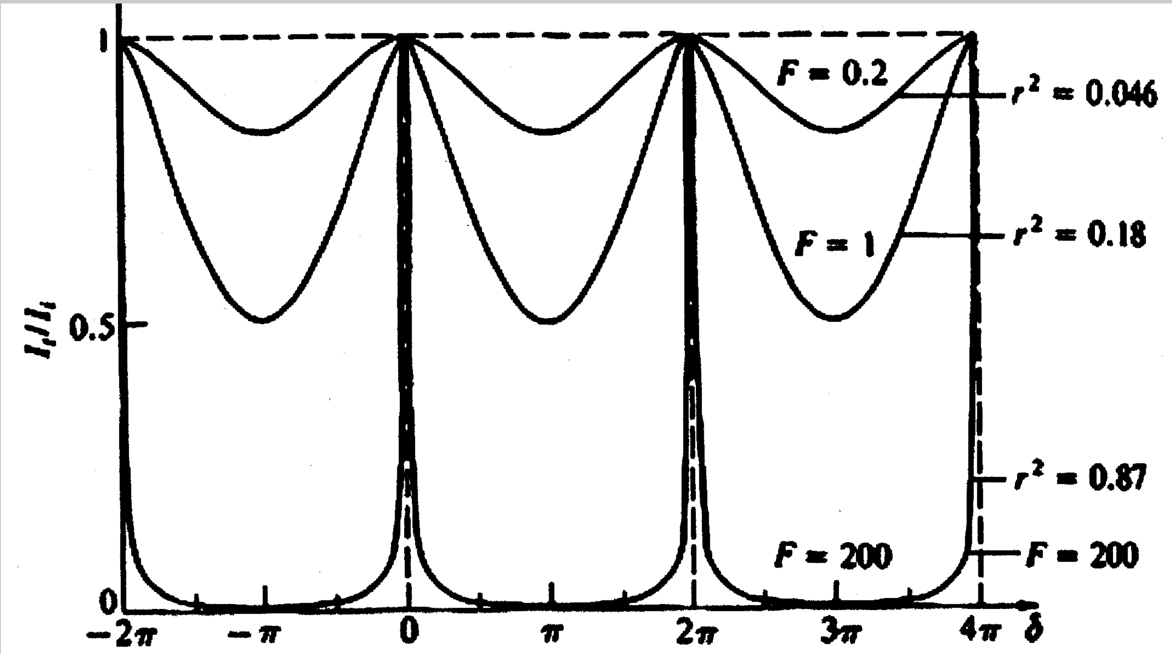
Destructive interference for reflected wave

Minimum transmission: sin( ) = 1, d = 2 π (m+1/2)



Constructive interference for reflected wave

Etalon transmittance vs. thickness, wavelength, or angle



$$T = \frac{1}{1 + F\sin^2(\delta/2)}$$

Transmission max:
sin( ) = 0, d = 2 π m

$$\delta = \frac{\omega}{c} 2 n L \cos[\theta_t]$$
$$= 2\pi m$$

At normal incidence:

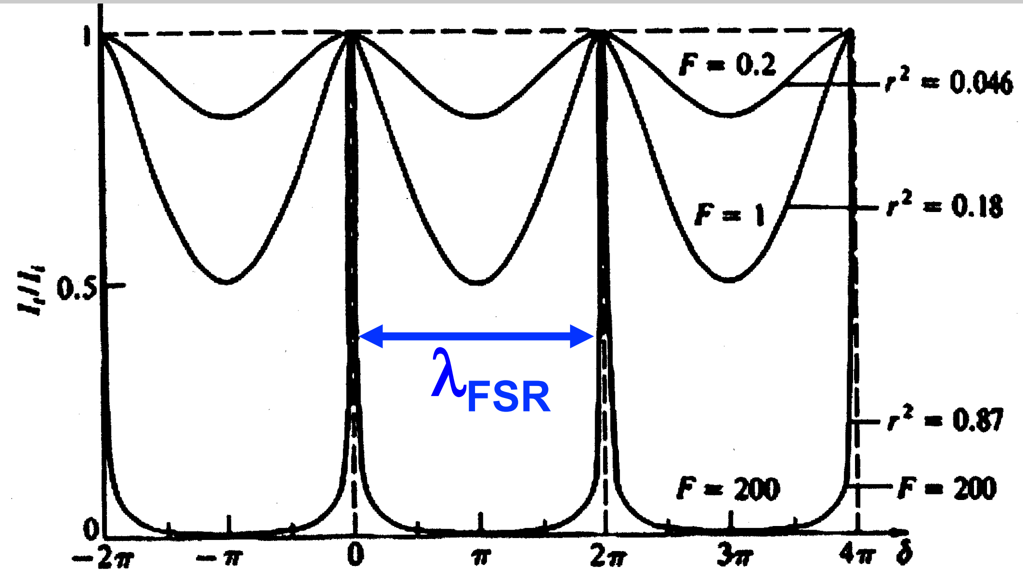
$$\lambda_m = \frac{2nL}{m} \quad \text{or} \quad nL = m\frac{\lambda_m}{2}$$

- The transmittance varies significantly with thickness or wavelength.
- We can also vary the incidence angle, which also affects δ.
- As the reflectance of each surface ($R=r^2$) approaches 1, the widths of the high-transmission regions become very narrow.

The Etalon Free Spectral Range

The Free Spectral Range is the wavelength range between transmission maxima.

λ_{FSR} = Free Spectral Range



For neighboring orders:

$$\frac{4\pi nL}{\lambda_1} - \frac{4\pi nL}{\lambda_2} = 2\pi \Rightarrow \frac{1}{\lambda_1} - \frac{1}{\lambda_2} = \frac{1}{2nL} = \frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2}$$

$$\lambda_2 - \lambda_1 = \lambda_{FSR}$$

$$\lambda_2 \lambda_1 \approx \lambda^2$$

$$\boxed{\lambda_{FSR} \approx \frac{\lambda^2}{2nL}}$$

$$\frac{\lambda_{FSR}}{\lambda} = \frac{\lambda}{2nL} = \frac{\nu_{FSR}}{\nu}$$

$$\boxed{\nu_{FSR} \approx \frac{c}{2nL}}$$

1/(round trip time)

Etalon Linewidth

The Linewidth δ_{LW} is a transmittance peak's full-width-half-max (FWHM).

$$T = \frac{1}{1 + F\sin^2(\delta/2)}$$

A maximum is where $\delta/2 \approx m\pi + \delta'/2$ and $\sin^2(\delta/2) \approx \delta'/2$

Under these conditions (near resonance),

$$T = \frac{1}{1 + F\delta'^2/4}$$

This is a Lorentzian profile, with FWHM at:

$$\frac{F}{4}\left(\frac{\delta_{LW}}{2}\right)^2 = 1 \quad \Rightarrow \quad \delta_{LW} \approx 4/\sqrt{F}$$

This transmission linewidth corresponds to the minimum resolvable wavelength.

Etalon Finesse

The Finesse, $\Im$, is the ratio of the
Free Spectral Range and the Linewidth:

δ = 2π corresponds
to one FSR

$$\Im \equiv \frac{\delta_{FSR}}{\delta_{FW}} = \frac{2\pi}{4/\sqrt{F}} = \frac{\pi\sqrt{F}}{2}$$

Using: $$F = \left[\frac{2r}{1-r^2}\right]^2$$

$$\boxed{\Im = \frac{\pi}{1-r^2}}$$

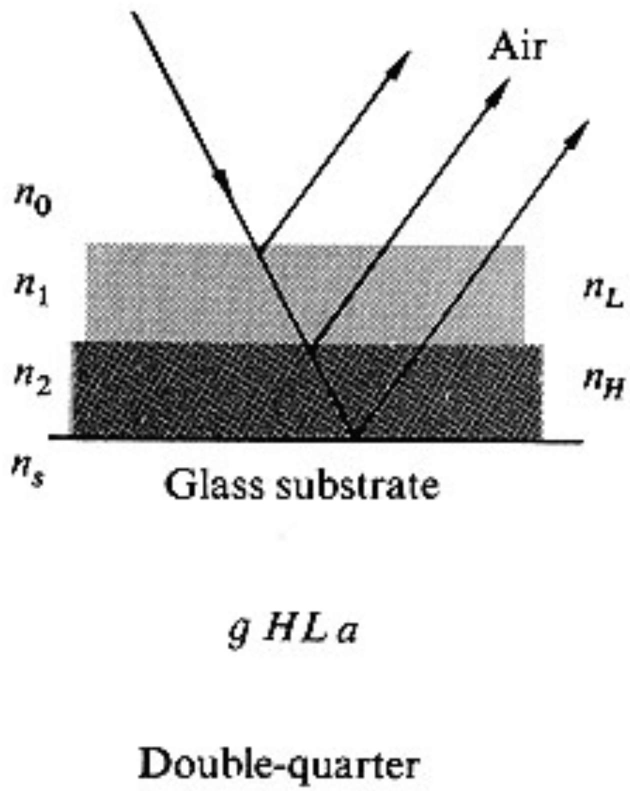
taking $r \approx 1$

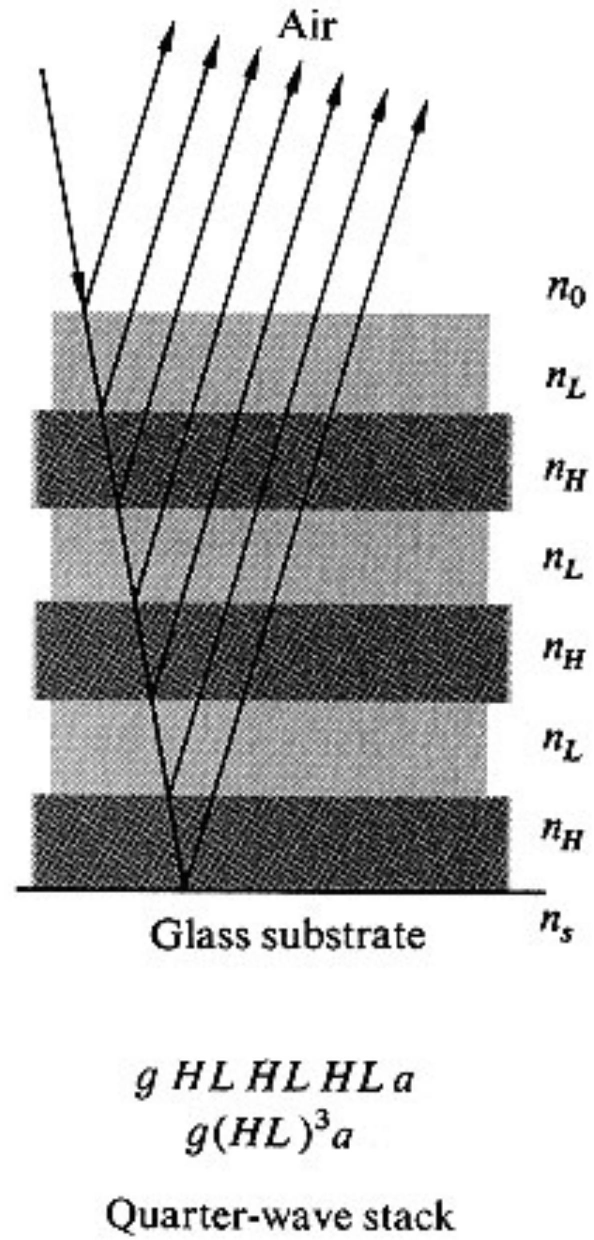
The Finesse is the number of wavelengths the interferometer can resolve.

Multilayer coatings

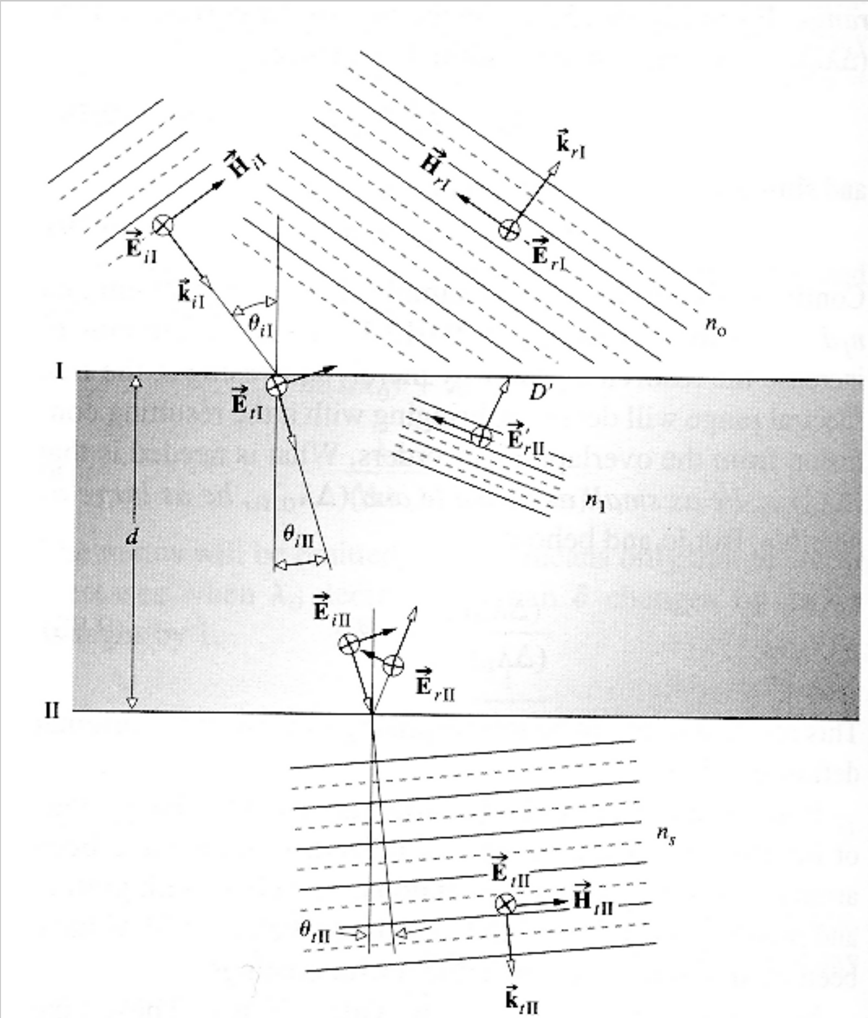
Typical laser mirrors and camera lenses use many layers.

The reflectance and transmittance can be custom designed



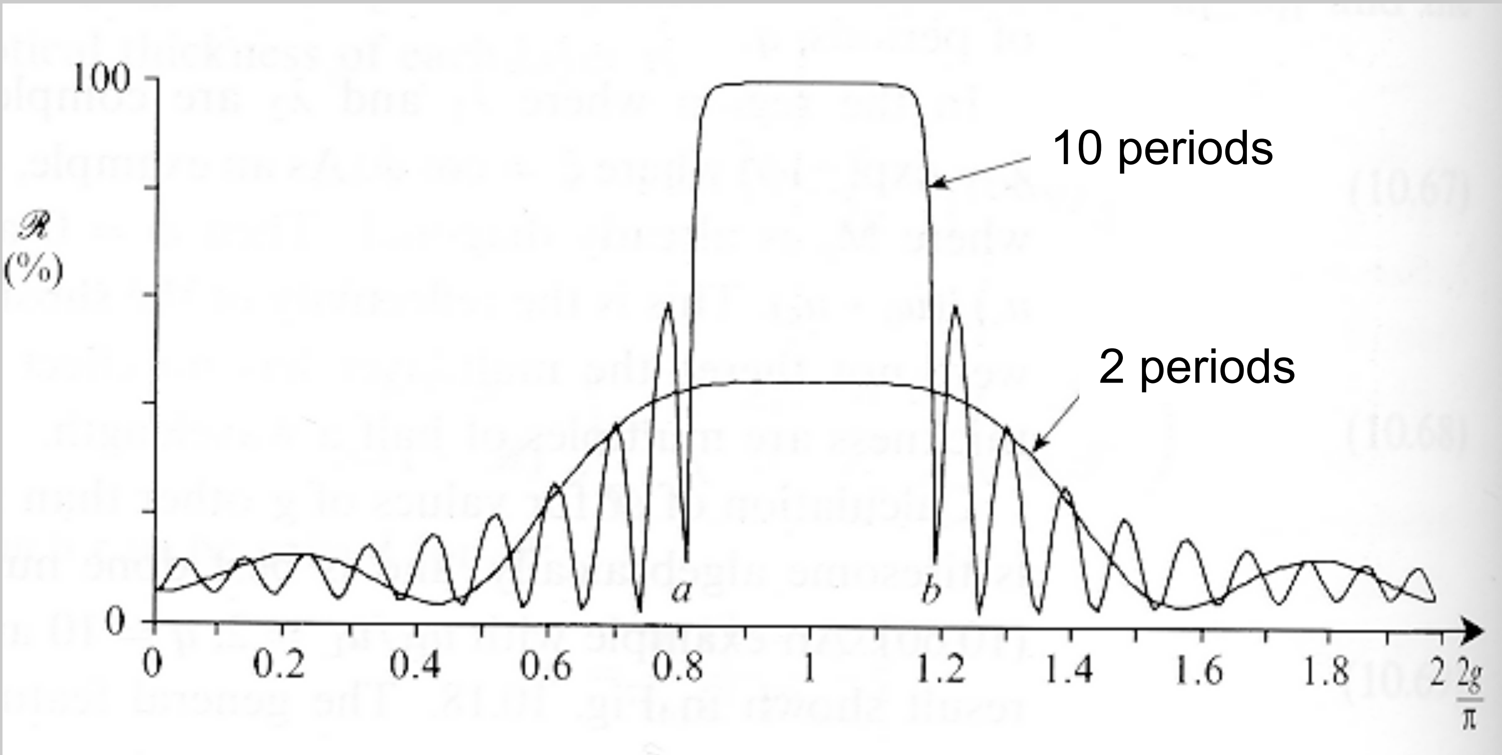


Multilayer thin-films: wave/matrix treatment



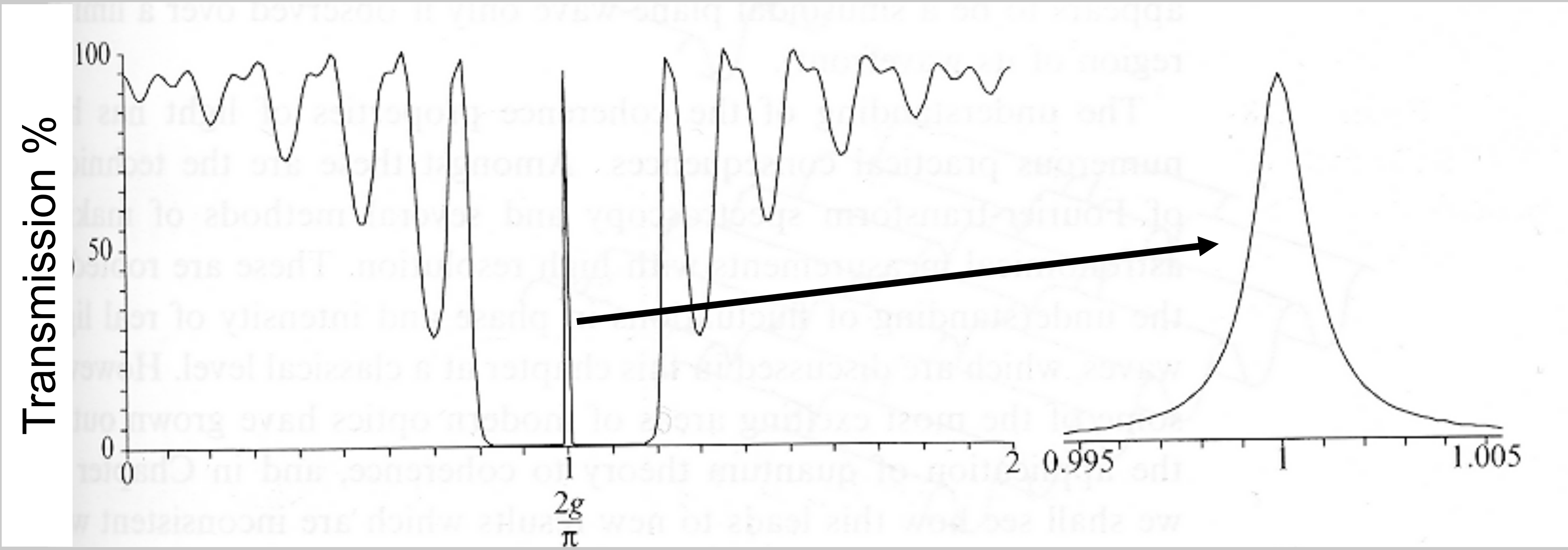
- Use boundary conditions to relate fields at the boundaries
- Phase shifts connect fields just after I to fields just before II
- Express this relation as a transfer matrix
- Multiply matrices for multiple layers

High-reflector design



Reflectivity can reach > 99.99% at a specific wavelength
> 99.5% for over 250nm
Bandwidth and reflectivity are better for “S” polarization.

Interference filter design



A thin layer is sandwiched between two high reflector coatings
-very large free spectral range, high finesse
- typically 5-10nm bandwidth, available throughout UV to IR