(a) the vector **E** in the incident wave is perpendicular to the plane of incidence (the xz-plane), i.e. it is parallel to the plane of the screen (the xy-plane). The sum of the fields in the incident and reflected waves at the surface of the screen is

$$E_0 = 0, \quad H_{0x} = 2H \cos\alpha = 2E \cos\alpha$$

(α being the angle of incidence). Hence

$$d\sigma = \frac{16a^6\omega^4}{9\pi^2c^4}\cos^2\alpha\,(1-\sin^2\vartheta\cos^2\phi)\,do,$$

where ϑ is the angle between the direction of diffraction **n** and the normal to the screen (the z-axis), and ϕ is the azimuth of the vector **n** with respect to the plane of incidence. The total cross-section is

$$\sigma = (64\omega^4a^6/27\pi c^4)\cos^2\alpha.$$

(b) the vector **E** lies in the plane of incidence. Then $E_0 = E_{0z} = -2E\sin\alpha$, $H_0 = H_{0y} = 2H = 2E$. The differential cross-section is

$$d\sigma = \frac{16a^6\omega^4}{9\pi^2c^4}\{\cos^2\vartheta + \sin^2\vartheta\,(\cos^2\phi + \tfrac{1}{4}\sin^2\alpha) - \sin\vartheta\sin\alpha\cos\phi\}\,do,$$

and the total cross-section is $\sigma = (64a^6\omega^4/27\pi c^4)(1+\frac{1}{4}\sin^2\alpha)$.

For natural incident light $\sigma = (64a^6\omega^4/27\pi c^4)(1-\frac{3}{8}\sin^2\alpha)$.

CHAPTER XI

ELECTROMAGNETIC WAVES IN ANISOTROPIC MEDIA

§96. The permittivity of crystals

THE properties of an anisotropic medium with respect to electromagnetic waves are defined by the tensors $\varepsilon_{ik}(\omega)$ and $\mu_{ik}(\omega)$, which give the relation between the inductions and the fields:†

$$D_i = \varepsilon_{ik}(\omega)E_k, \quad B_i = \mu_{ik}(\omega)H_k. \tag{96.1}$$

In what follows we shall, for definiteness, consider the electric field and the tensor ε_{ik}; all the results obtained are valid for the tensor μ_{ik} also.

As $\omega \to 0$, the ε_{ik} tend to their static values, which have been shown in §13 to be symmetrical with respect to i and k. The proof was thermodynamical, and therefore holds only for states of thermodynamic equilibrium. In a variable field, a substance is of course not in equilibrium, and the proof in §13 is consequently invalid. To ascertain the properties of the tensor ε_{ik} we must use the generalized principle of the symmetry of the kinetic coefficients (see *SP* 1, §125).

The generalized susceptibilities $\alpha_{ab}(\omega)$ which appear in the formulation of this principle are defined in terms of the response of the system to a perturbation:

$$\hat{V} = -\hat{x}_a f_a(t)$$

(where the x_a are quantities describing the system) and are the coefficients in the linear relation between the Fourier components of the mean values $\overline{x_a}(t)$ and the generalized forces $f_a(t)$:

$$\overline{x}_{a\omega} = \alpha_{ab}(\omega) f_{b\omega}.$$

The change in the energy of the system with time under the perturbation is given by

$$\dot{\mathscr{U}} = -\dot{f}_a\overline{x_a}.$$

According to the symmetry principle,

$$\alpha_{ab}(\omega) = \alpha_{ba}(\omega),$$

if the system is not in an external magnetic field and has no magnetic structure; otherwise, $\alpha_{ba}(\omega)$ has to be taken for the "time-reversed" system.

It is easy to relate the components of the tensor $\varepsilon_{ik}(\omega)$ to the generalized susceptibilities. To do so, we note that the rate of change of the energy of a dielectric body in a variable

† It should be recalled that these quantities refer to the variable fields in the wave; the possible presence of a constant induction (in a pyroelectric or ferromagnetic crystal) is irrelevant to this discussion.

electric field is given by the integral

$$\int \frac{1}{4\pi}\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}\,dV. \tag{96.2}$$

A comparison with the above formulae shows that, if the components of the vector **E** at each point are taken as the quantities x_a, the corresponding quantities f_a will be the components of **D**. (The suffix a takes a continuous series of values, labelling both the components of the vectors and the points in the body.) The coefficients α_{ab} are then the components of the tensor $\varepsilon^{-1}{}_{ik}$. The symmetry properties of ε_{ik} are, of course, identical with those of its inverse. Since **E** and $\dot{\mathbf{D}}$ are multiplied in (96.2) only at the same point in the body, the interchange of the suffixes a and b is equivalent to simply interchanging the tensor suffixes. We thus conclude that the tensor ε_{ik} is symmetrical:†

$$\varepsilon_{ik}(\omega) = \varepsilon_{ki}(\omega). \tag{96.3}$$

It should be noted that the components of the polarizability tensor for the whole body, i.e. the coefficients in the equations $\mathscr{P}_i = V\alpha_{ik}\mathfrak{E}_k$, also come under the definition of generalized susceptibilities. For the rate of change of the energy of a body placed in a variable external field $\mathfrak{E}$ is

$$-\mathscr{P}\cdot d\mathfrak{E}/dt. \tag{96.4}$$

Hence we see that, if the x_a are the three components of the vector $\mathscr{P}$, then the corresponding f_a are those of the vector $\mathfrak{E}$, so that the coefficients α_{ab} are in this case $V\alpha_{ik}$.

Several of the formulae derived previously for an isotropic medium can be directly generalized to the anisotropic case. Repeating for the anisotropic case the derivation in §80, we find that the energy dissipation in a monochromatic electromagnetic field is

$$Q = \frac{i\omega}{16\pi}\left\{\left(\varepsilon_{ik}{}^* - \varepsilon_{ki}\right)E_iE_k{}^* + \left(\mu_{ik}{}^* - \mu_{ki}\right)H_iH_k{}^*\right\}, \tag{96.5}$$

which is analogous to (80.5). The condition that absorption be absent is $\varepsilon_{ik}{}^* = \varepsilon_{ki} = \varepsilon_{ik}$, i.e. the ε_{ik} must be real, as must the μ_{ik}.

When absorption is absent, the internal electromagnetic energy per unit volume can be defined as shown in §80. The formula for an anisotropic medium corresponding to (80.11) is

$$\bar{U} = \frac{1}{16\pi}\left\{\frac{d}{d\omega}(\omega\varepsilon_{ik})E_iE_k{}^* + \frac{d}{d\omega}(\omega\mu_{ik})H_iH_k{}^*\right\}. \tag{96.6}$$

In §87 we used the surface impedance ζ, in terms of which the boundary conditions at the surface of a metal can be formulated even if the permittivity is no longer meaningful. At the surface of an anisotropic body the boundary condition corresponding to (87.6) is

$$E_\alpha = \zeta_{\alpha\beta}(\mathbf{H}\times\mathbf{n})_\beta, \tag{96.7}$$

where $\zeta_{\alpha\beta}(\omega)$ is a two-dimensional tensor on the surface of the body. It should be borne in mind that the value of this tensor depends, in general, on the crystallographic direction of the surface concerned.

† The properties of this tensor in the presence of an external magnetic field will be discussed in §101.

The energy flux into the body is $(c/4\pi)\mathbf{E}\times\mathbf{H}\cdot\mathbf{n} = (c/4\pi)\mathbf{E}\cdot\mathbf{H}\times\mathbf{n} \equiv (c/4\pi)E_\alpha(\mathbf{H}\times\mathbf{n})_\alpha$. (Here **E** and **H** are real.) Hence we see that if, in applying the principle of the symmetry of the kinetic coefficients, we take the components E_α as the x_a, then the corresponding f_a will be $-(\mathbf{H}\times\mathbf{n})_\alpha$, i.e. f_a will be $-(i/\omega)(\mathbf{H}\times\mathbf{n})_\alpha$ (returning to the complex form). The coefficients α_{ab} are therefore the same, apart from a factor, as the components $\zeta_{\alpha\beta}$, and we conclude that

$$\zeta_{\alpha\beta} = \zeta_{\beta\alpha} \tag{96.8}$$

in the absence of an external magnetic field.

PROBLEM

Express the components of the tensor $\zeta_{\alpha\beta}$ in terms of those of $\eta_{\alpha\beta} \equiv \varepsilon^{-1}{}_{\alpha\beta}$, assuming that the latter exists and that the body is non-magnetic ($\mu_{ik} = \delta_{ik}$).

SOLUTION. In an anisotropic medium, the equation $\zeta^2 = 1/\varepsilon$ (87.2) becomes $\zeta_{\alpha\gamma}\zeta_{\gamma\beta} = \eta_{\alpha\beta}$. In components this gives

$$\zeta_{11}{}^2 + \zeta_{12}\zeta_{21} = \eta_{11}, \quad \zeta_{22}{}^2 + \zeta_{12}\zeta_{21} = \eta_{22},$$
$$\zeta_{12}(\zeta_{11}+\zeta_{22}) = \eta_{12}, \quad \zeta_{21}(\zeta_{11}+\zeta_{22}) = \eta_{21}.$$

The solution of these equations is

$$\zeta_{12} = \eta_{12}/\xi, \quad \zeta_{21} = \eta_{21}/\xi,$$
$$\zeta_{11} = [\eta_{11} \pm \sqrt{(\eta_{11}\eta_{22} - \eta_{12}\eta_{21})}]/\xi, \quad \zeta_{22} = [\eta_{22} \pm \sqrt{(\eta_{11}\eta_{22} - \eta_{12}\eta_{21})}]/\xi,$$
$$\xi^2 = \eta_{11} + \eta_{22} \pm 2\sqrt{(\eta_{11}\eta_{22} - \eta_{12}\eta_{21})}.$$

The choice of signs is determined by the condition that the absorption of energy must be positive. We do not assume $\zeta_{12} = \zeta_{21}$, and thereby allow for the presence of an external magnetic field.

§97. A plane wave in an anisotropic medium

In studying the optics of anisotropic bodies (crystals) we shall take only the most important case, where the medium may be supposed non-magnetic and transparent in a given range of frequencies. Accordingly, the relation between the electric and magnetic fields and inductions is

$$D_i = \varepsilon_{ik}E_k, \quad \mathbf{B} = \mathbf{H}. \tag{97.1}$$

The components of the dielectric tensor ε_{ik} are all real, and its principal values are positive.

Maxwell's equations for the field of a monochromatic wave with frequency ω are

$$i\omega\mathbf{H} = c\,\mathbf{curl}\,\mathbf{E}, \quad i\omega\mathbf{D} = -c\,\mathbf{curl}\,\mathbf{H}. \tag{97.2}$$

In a plane wave propagated in a transparent medium all quantities are proportional to $e^{i\mathbf{k}\cdot\mathbf{r}}$, with a real wave vector **k**. Effecting the differentiation with respect to the coordinates, we obtain

$$\omega\mathbf{H}/c = \mathbf{k}\times\mathbf{E}, \quad \omega\mathbf{D}/c = -\mathbf{k}\times\mathbf{H}. \tag{97.3}$$

Hence we see, first of all, that the three vectors **k**, **D**, **H** are mutually perpendicular. Moreover, **H** is perpendicular to **E**, and so the three vectors **D**, **E**, **k**, being all perpendicular to **H**, must be coplanar. Fig. 51 (p. 334) shows the relative position of all these vectors. With respect to the direction of the wave vector, **D** and **H** are transverse, but **E** is not. The diagram shows also the direction of the energy flux **S** in the wave. It is given by the vector product **E** × **H**, i.e. it is perpendicular to both **E** and **H**. The direction of **S** is not the same as

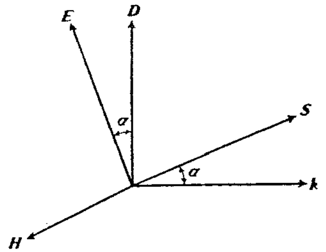

FIG. 51

that of **k**, unlike what happens for an isotropic medium. Clearly the vector **S** is coplanar with **E**, **D** and **k**, and the angle between **S** and **k** is equal to that between **E** and **D**.

We can define a vector **n** by

$$\mathbf{k} = \omega\mathbf{n}/c. \tag{97.4}$$

The magnitude of this vector in an anisotropic medium depends on its direction, whereas in an isotropic medium $n = \sqrt{\varepsilon}$ depends only on the frequency.† Using (97.4), we can write the fundamental formulae (97.3) as

$$\mathbf{H} = \mathbf{n}\times\mathbf{E}, \qquad \mathbf{D} = -\mathbf{n}\times\mathbf{H}. \tag{97.5}$$

The energy flux vector in a plane wave is

$$\mathbf{S} = c\mathbf{E}\times\mathbf{H}/4\pi = (c/4\pi)\{E^2\mathbf{n} - (\mathbf{E}\cdot\mathbf{n})\mathbf{E}\}; \tag{97.6}$$

in this formula **E** and **H** are real.

So far we have not used the relation (97.1) which involves the constants ε_{ik} characterizing the material. This relation, together with equations (97.5), determines the function $\omega(\mathbf{k})$.

Substituting the first equation (97.5) in the second, we have

$$\mathbf{D} = \mathbf{n}\times(\mathbf{E}\times\mathbf{n}) = n^2\mathbf{E} - (\mathbf{n}\cdot\mathbf{E})\mathbf{n}. \tag{97.7}$$

If we equate the components of this vector to $\varepsilon_{ik}E_k$ in accordance with (97.1), we obtain three linear homogeneous equations for the three components of **E**: $n^2E_i - n_in_kE_k = \varepsilon_{ik}E_k$ or

$$(n^2\delta_{ik} - n_in_k - \varepsilon_{ik})E_k = 0 \tag{97.8}$$

The compatibility condition for these equations is that the determinant of their coefficients should vanish:

$$\det|n^2\delta_{ik} - n_in_k - \varepsilon_{ik}| = 0. \tag{97.9}$$

In practice, this determinant is conveniently evaluated by taking as the axes of x, y, z the principal axes of the tensor ε_{ik} (called the *principal dielectric axes*). Let the principal values of the tensor be $\varepsilon^{(x)}$, $\varepsilon^{(y)}$, $\varepsilon^{(z)}$. Then a simple calculation gives

$$n^2(\varepsilon^{(x)}n_x{}^2 + \varepsilon^{(y)}n_y{}^2 + \varepsilon^{(z)}n_z{}^2) - [n_x{}^2\varepsilon^{(x)}(\varepsilon^{(y)} + \varepsilon^{(z)}) + n_y{}^2\varepsilon^{(y)}(\varepsilon^{(x)} + \varepsilon^{(z)}) + n_z{}^2\varepsilon^{(z)}(\varepsilon^{(x)} + \varepsilon^{(y)})] + \varepsilon^{(x)}\varepsilon^{(y)}\varepsilon^{(z)} = 0. \tag{97.10}$$

† The magnitude n is still called the *refractive index*, although it no longer bears the same simple relation to the law of refraction as in isotropic bodies.

The sixth-order terms cancel when the determinant is expanded; this is, of course, no accident and is due ultimately to the fact that the wave has two, not three, independent directions of polarization.

Equation (97.10), called *Fresnel's equation*, is one of the fundamental equations of crystal optics.† It determines implicitly the dispersion relation, i.e. the frequency as a function of the wave vector. (The principal values $\varepsilon^{(i)}$ are functions of frequency, and so are, in some cases (see §99), the directions of the principal axes of the tensor ε_{ik}.) For monochromatic waves, however, ω, and therefore all the $\varepsilon^{(i)}$, are usually given constants, and equation (97.10) then gives the magnitude of the wave vector as a function of its direction. When the direction of **n** is given, (97.10) is a quadratic equation, for n^2, with real coefficients. Hence two different magnitudes of the wave vector correspond, in general, to each direction of **n**.

Equation (97.10) (with constant coefficients $\varepsilon^{(i)}$) defines in the coordinates n_x, n_y, n_z the "wave-vector surface".‡ In general this is a surface of the fourth order, whose properties will be discussed in detail in the following sections. Here we shall mention some general properties of this surface.

We first introduce another quantity charactèrizing the propagation of light in an anisotropic medium. The direction of the light rays (in geometrical optics) is given by the group velocity vector $\partial\omega/\partial\mathbf{k}$. In an isotropic medium, the direction of this vector is always the same as that of the wave vector, but in an anisotropic medium the two do not in general coincide. The rays may be characterized by a vector **s**, whose direction is that of the group velocity, while its magnitude is given by

$$\mathbf{n}\cdot\mathbf{s} = 1. \tag{97.11}$$

We shall call **s** the *ray vector*. Its significance is as follows.

Let us consider a beam of rays (of a single frequency) propagated in all directions from some point. The value of the eikonal ψ (which is, apart from a factor ω/c, the wave phase; see §85) at any point is given by the integral $\int\mathbf{n}\cdot d\mathbf{l}$ taken along the ray. Using the vector **s** which determines the direction of the ray, we can put

$$\psi = \int\mathbf{n}\cdot d\mathbf{l} = \int(\mathbf{n}\cdot\mathbf{s}/s)\,dl = \int dl/s. \tag{97.12}$$

In a homogeneous medium, s is constant along the ray, so that $\psi = L/s$, where L is the length of the ray segment concerned. Hence we see that, if a segment equal (or proportional) to s is taken along each ray from the centre, the resulting surface is such that the phase of the rays is the same at every point. This is called the *ray surface*.

The wave-vector surface and the ray surface are in a certain dual relationship. Let the equation of the wave-vector surface be written $f(\omega, \mathbf{k}) = 0$. Then the group velocity vector is

$$\frac{\partial\omega}{\partial\mathbf{k}} = -\frac{\partial f/\partial\mathbf{k}}{\partial f/\partial\omega}, \tag{97.13}$$

i.e. is proportional to $\partial f/\partial\mathbf{k}$, or, what is the same thing (since the derivative is taken for constant ω), to $\partial f/\partial\mathbf{n}$. The ray vector, therefore, is also proportional to $\partial f/\partial\mathbf{n}$. But the vector

† The foundations of crystal optics were laid by A. J. Fresnel in the 1820s, on the basis of mechanical analogies, long before the development of the electromagnetic theory.

‡ A concept called the "surface of normals" or "surface of indices" has been used; it is obtained by taking a point at a distance $1/n$ (instead of n) in each direction, but is less convenient.

$\partial f/\partial \mathbf{n}$ is normal to the surface $f = 0$. Thus we conclude that the direction of the ray vector of a wave with given $\mathbf{n}$ is that of the normal at the corresponding point of the wave-vector surface.

It is easy to see that the reverse is also true: the normal to the ray surface gives the direction of the corresponding wave vectors. For the equation $\mathbf{s}\cdot\delta\mathbf{n} = 0$, where $\delta\mathbf{n}$ is an arbitrary infinitesimal change in $\mathbf{n}$ (for given ω), i.e. the vector of an infinitesimal displacement on the surface, expresses the fact that $\mathbf{s}$ is perpendicular to the wave-vector surface. Differentiating (again for given ω) the equation $\mathbf{n}\cdot\mathbf{s} = 1$, we obtain $\mathbf{n}\cdot\delta\mathbf{s}+\mathbf{s}\cdot\delta\mathbf{n} = 0$, and therefore $\mathbf{n}\cdot\delta\mathbf{s} = 0$, which proves the above statement.

This relation between the surfaces of $\mathbf{n}$ and $\mathbf{s}$ can be made more precise. Let $\mathbf{n}_0$ be the position vector of a point on the wave-vector surface, and $\mathbf{s}_0$ the corresponding ray vector. The equation (in coordinates n_x, n_y, n_z) of the tangent plane at this point is $\mathbf{s}_0\cdot(\mathbf{n}-\mathbf{n}_0) = 0$, which states that $\mathbf{s}_0$ is perpendicular to any vector $\mathbf{n}-\mathbf{n}_0$ in the plane. Since $\mathbf{s}_0$ and $\mathbf{n}_0$ are related by $\mathbf{s}_0\cdot\mathbf{n}_0 = 1$, we can write the equation as

$$\mathbf{s}_0\cdot\mathbf{n} = 1. \qquad (97.14)$$

Hence it follows that $1/s_0$ is the length of the perpendicular from the origin to the tangent plane to the wave-vector surface at the point $\mathbf{n}_0$.

Conversely, the length of the perpendicular from the origin to the tangent plane to the ray surface at a point $\mathbf{s}_0$ is $1/n_0$.

To ascertain the location of the ray vector relative to the field vectors in the wave, we notice that the group velocity is always in the same direction as the (time) averaged energy flux vector. For let us consider a wave packet, occupying a small region of space. When the packet moves, the energy concentrated in it must move with it, and the direction of the energy flux is therefore the same as the direction of the velocity of the packet, i.e. the group velocity. It can be demonstrated from (97.5) that the group velocity is in the same direction as the Poynting vector. Differentiating (for given ω), we obtain

$$\delta\mathbf{D} = \delta\mathbf{H}\times\mathbf{n}+\mathbf{H}\times\delta\mathbf{n}, \qquad \delta\mathbf{H} = \mathbf{n}\times\delta\mathbf{E}+\delta\mathbf{n}\times\mathbf{E}. \qquad (97.15)$$

We take the scalar product of the first equation with $\mathbf{E}$ and of the second with $\mathbf{H}$, obtaining

$$\mathbf{E}\cdot\delta\mathbf{D} = \mathbf{H}\cdot\delta\mathbf{H}+\mathbf{E}\times\mathbf{H}\cdot\delta\mathbf{n}, \qquad \mathbf{H}\cdot\delta\mathbf{H} = \mathbf{D}\cdot\delta\mathbf{E}+\mathbf{E}\times\mathbf{H}\cdot\delta\mathbf{n}.$$

But $\mathbf{D}\cdot\delta\mathbf{E} = \varepsilon_{ik}E_k\delta E_i = \mathbf{E}\cdot\delta\mathbf{D}$, and so, adding the two equations, we have

$$\mathbf{E}\times\mathbf{H}\cdot\delta\mathbf{n} = 0, \qquad (97.16)$$

i.e. the vector $\mathbf{E}\times\mathbf{H}$ is normal to the wave-vector surface. This is the required result.†

Since the Poynting vector is perpendicular to $\mathbf{H}$ and $\mathbf{E}$, the same is true of $\mathbf{s}$:

$$\mathbf{s}\cdot\mathbf{H} = 0, \qquad \mathbf{s}\cdot\mathbf{E} = 0. \qquad (97.17)$$

A direct calculation, using formulae (97.5), (97.11) and (97.17), gives

$$\mathbf{H} = \mathbf{s}\times\mathbf{D}, \qquad \mathbf{E} = -\mathbf{s}\times\mathbf{H}. \qquad (97.18)$$

For example, $\mathbf{s}\times\mathbf{H} = \mathbf{s}\times(\mathbf{n}\times\mathbf{E}) = \mathbf{n}(\mathbf{s}\cdot\mathbf{E})-\mathbf{E}(\mathbf{n}\cdot\mathbf{s}) = -\mathbf{E}$.

† The result thus obtained relates to the instantaneous, as well as to the average, energy flux. In this proof, however, the symmetry of the tensor ε_{ik} is vital. The result is therefore not valid in the above form for media in which ε_{ik} is not symmetrical (*gyrotropic* media, §101). The statement is still valid, however, for the average value of the Poynting vector (§101, Problem 1).

If we compare formulae (97.18) and (97.5), we see that they differ by the interchange of

$$\mathbf{E} \text{ and } \mathbf{D}, \qquad \mathbf{n} \text{ and } \mathbf{s}, \qquad \varepsilon_{ik} \text{ and } \varepsilon^{-1}{}_{ik} \qquad (97.19)$$

(the relation $\mathbf{n}\cdot\mathbf{s} = 1$ remaining valid, of course). The last of these pairs must be included in order that the relation (97.1) between $\mathbf{D}$ and $\mathbf{E}$ should remain valid. Thus the following useful rule may be formulated: an equation valid for one set of quantities can be converted into one valid for another set by means of the interchanges (97.19).

In particular, the application of this rule to (97.10) gives immediately an analogous equation for $\mathbf{s}$:

$$s^2(\varepsilon^{(y)}\varepsilon^{(z)}s_x^{\ 2}+\varepsilon^{(x)}\varepsilon^{(z)}s_y^{\ 2}+\varepsilon^{(x)}\varepsilon^{(y)}s_z^{\ 2})$$
$$-[s_x^{\ 2}(\varepsilon^{(y)}+\varepsilon^{(z)})+s_y^{\ 2}(\varepsilon^{(x)}+\varepsilon^{(z)})+s_z^{\ 2}(\varepsilon^{(x)}+\varepsilon^{(y)})]+1 = 0. \qquad (97.20)$$

This equation gives the form of the ray surface. Like the wave-vector surface, it is of the fourth order. When the direction of $\mathbf{s}$ is given, (97.20) is a quadratic equation for s^2, which in general has two different real roots. Thus two rays with different wave vectors can be propagated in any direction in the crystal.

Let us now consider the polarization of waves propagated in an anisotropic medium. Equations (97.8), from which Fresnel's equation has been derived, are unsuitable for this, because they involve the field $\mathbf{E}$, whereas it is the induction $\mathbf{D}$ which is transverse (to the given $\mathbf{n}$) in the wave. In order to take account immediately of the fact that $\mathbf{D}$ is transverse, we use for the time being a new coordinate system with one axis in the direction of the wave vector, and denote the two transverse axes by Greek suffixes, which take the values 1 and 2. The transverse components of equation (97.7) give $D_\alpha = n^2E_\alpha$; substituting $E_\alpha = \varepsilon^{-1}{}_{\alpha\beta}D_\beta$, where $\varepsilon^{-1}{}_{\alpha\beta}$ is a component of the tensor inverse to $\varepsilon_{\alpha\beta}$, we have

$$\left(\frac{1}{n^2}\delta_{\alpha\beta}-\varepsilon^{-1}{}_{\alpha\beta}\right)D_\beta = 0. \qquad (97.21)$$

The condition for the two equations ($\alpha = 1, 2$) in the two unknowns D_1, D_2 to be compatible is that their determinant should be zero:

$$\det|n^{-2}\delta_{\alpha\beta}-\varepsilon^{-1}{}_{\alpha\beta}| = 0. \qquad (97.22)$$

This condition is, of course, the same as Fresnel's equation, which was written in the original coordinates x, y, z. We now see also, however, that the vectors $\mathbf{D}$ corresponding to the two values of n are along the principal axes of the symmetrical two-dimensional tensor of rank two $\varepsilon^{-1}{}_{\alpha\beta}$. According to general theorems it follows that these two vectors are perpendicular. Thus, in the two waves with the wave vector in the same direction, the electric induction vectors are linearly polarized in two perpendicular planes.

Equations (97.21) have a simple geometrical interpretation. Let us draw the tensor ellipsoid corresponding to the tensor $\varepsilon^{-1}{}_{ik}$, returning to the principal dielectric axes, i.e. the surface

$$\varepsilon^{-1}{}_{ik}x_ix_k = \frac{x^2}{\varepsilon^{(x)}}+\frac{y^2}{\varepsilon^{(y)}}+\frac{z^2}{\varepsilon^{(z)}} = 1 \qquad (97.23)$$

(Fig. 52, p. 338). Let this ellipsoid be cut by a plane through its centre perpendicular to the given direction of $\mathbf{n}$. The section is in general an ellipse; the lengths of its axes determine the values of n, and their directions determine the directions of the oscillations, i.e. the vectors $\mathbf{D}$.

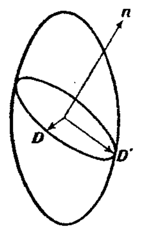

FIG. 52

From this construction (with, in general, $\varepsilon^{(x)}, \varepsilon^{(y)}, \varepsilon^{(z)}$ different) we see at once that, if the wave vector is in (say) the x-direction, the directions of polarization (**D**) will be the y and z directions. If the vector **n** lies in one of the coordinate planes, e.g. the xy-plane, one of the directions of polarization is also in that plane, and the other is in the z-direction.

The polarizations of two waves with the ray vector in the same direction have similar properties. Instead of the directions of the induction **D**, we must now consider those of the vector **E**, which is transverse to **s**, and equations (97.21) are replaced by the analogous equations

$$\left(\frac{1}{s^2}\delta_{\alpha\beta} - \varepsilon_{\alpha\beta}\right)E_\beta = 0. \tag{97.24}$$

The geometrical construction is here based on the tensor ellipsoid

$$\varepsilon_{ik}x_i x_k = \varepsilon^{(x)}x^2 + \varepsilon^{(y)}y^2 + \varepsilon^{(z)}z^2 = 1, \tag{97.25}$$

corresponding to the tensor ε_{ik} itself (called the *Fresnel ellipsoid*).

It should be emphasized that plane waves propagated in an anisotropic medium are linearly polarized in certain planes. In this respect the optical properties of anisotropic media are very different from those of isotropic media. A plane wave propagated in an isotropic medium is in general elliptically polarized, and is linearly polarized only in particular cases. This important difference arises because the case of complete isotropy of the medium is in a sense one of degeneracy, in which a single wave vector corresponds to two directions of polarization, whereas in an anisotropic medium there are in general two different wave vectors (in the same direction). The two linearly polarized waves propagated with the same value of **n** combine to form one elliptically polarized wave.

PROBLEM

Express the components of the ray vector **s** in terms of the components of **n** along the principal dielectric axes.

SOLUTION. Differentiating the left-hand side of the equation $f(\mathbf{n}) = 0$ (97.10) with respect to n_i and determining from the condition $\mathbf{n}\cdot\mathbf{s} = 1$ the proportionality coefficient between s_i and $\partial f/\partial n_i$, we obtain the following relations between the vectors **s** and **n**:

$$\frac{s_x}{n_x} = \frac{\varepsilon^{(x)}(\varepsilon^{(y)}+\varepsilon^{(z)}) - 2\varepsilon^{(x)}n_x^2 - (\varepsilon^{(x)}+\varepsilon^{(y)})n_y^2 - (\varepsilon^{(x)}+\varepsilon^{(z)})n_z^2}{2\varepsilon^{(x)}\varepsilon^{(y)}\varepsilon^{(z)} - n_x^2\varepsilon^{(x)}(\varepsilon^{(y)}+\varepsilon^{(z)}) - n_y^2\varepsilon^{(y)}(\varepsilon^{(x)}+\varepsilon^{(z)}) - n_z^2\varepsilon^{(z)}(\varepsilon^{(x)}+\varepsilon^{(y)})}$$

and similarly for s_y, s_z.

§98. Optical properties of uniaxial crystals

The optical properties of a crystal depend primarily on the symmetry of its dielectric tensor ε_{ik}. In this respect all crystals fall under three types: cubic, uniaxial and biaxial (see §13). In a crystal of the cubic system $\varepsilon_{ik} = \varepsilon\delta_{ik}$, i.e. the three principal values of the tensor are equal, and the directions of the principal axes are arbitrary. As regards their optical properties, therefore, cubic crystals are no different from isotropic bodies.

The uniaxial crystals include those of the rhombohedral, tetragonal and hexagonal systems. Here one of the principal axes of the tensor ε_{ik} coincides with the threefold, fourfold or sixfold axis of symmetry respectively; in optics, this axis is called the *optical axis* of the crystal, and in what follows we shall take it as the z-axis, denoting the corresponding principal value of ε_{ik} by $\varepsilon_\parallel$. The directions of the other two principal axes, in a plane perpendicular to the optical axis, are arbitrary, and the corresponding principal values, which we denote by $\varepsilon_\perp$, are equal.

If in Fresnel's equation (97.10) we put $\varepsilon^{(x)} = \varepsilon^{(y)} = \varepsilon_\perp$, $\varepsilon^{(z)} = \varepsilon_\parallel$, the left-hand side is a product of two quadratic factors:

$$(n^2 - \varepsilon_\perp)[\varepsilon_\parallel n_z^2 + \varepsilon_\perp(n_x^2 + n_y^2) - \varepsilon_\perp\varepsilon_\parallel] = 0.$$

In other words, the quartic equation gives the two quadratic equations

$$n^2 = \varepsilon_\perp, \tag{98.1}$$

$$\frac{n_z^2}{\varepsilon_\perp} + \frac{n_x^2 + n_y^2}{\varepsilon_\parallel} = 1. \tag{98.2}$$

Geometrically, this signifies that the wave-vector surface, which is in general of the fourth order, becomes two separate surfaces, a sphere and an ellipsoid. Fig. 53 shows a cross-section of these surfaces. Two cases are possible: if $\varepsilon_\perp > \varepsilon_\parallel$, the sphere lies outside the ellipsoid, but if $\varepsilon_\perp < \varepsilon_\parallel$ it lies inside. In the first case we speak of a *negative uniaxial crystal*, and in the second case of a *positive* one. The two surfaces touch at opposite poles on the n_z-axis. The direction of the optical axis therefore corresponds to only one value of the wave vector.

The ray surface is similar in form. By the rule (97.19), its equation is obtained from (98.1) and (98.2):

$$s^2 = 1/\varepsilon_\perp, \tag{98.3}$$

$$\varepsilon_\perp s_z^2 + \varepsilon_\parallel(s_x^2 + s_y^2) = 1. \tag{98.4}$$

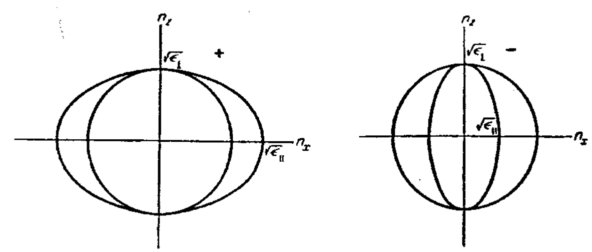

FIG. 53

In a positive crystal the ellipsoid lies within the sphere, and in a negative one outside.

Thus we see that two types of wave can be propagated in a uniaxial crystal. With respect to one type, called *ordinary waves*, the crystal behaves like an isotropic body with refractive index $n = \sqrt{\varepsilon_\perp}$. The magnitude of the wave vector is $\omega n/c$ whatever its direction, and the direction of the ray vector is that of $\mathbf{n}$.

In waves of the second type, called *extraordinary waves*, the magnitude of the wave vector depends on the angle θ which it makes with the optical axis. By (98.2)

$$\frac{1}{n^2} = \frac{\sin^2\theta}{\varepsilon_\parallel} + \frac{\cos^2\theta}{\varepsilon_\perp}. \tag{98.5}$$

The ray vector in an extraordinary wave is not in the same direction as the wave vector, but is coplanar with that vector and the optical axis, their common plane being called the *principal section* for the given $\mathbf{n}$. Let this be the zx-plane; the ratio of the derivatives of the left-hand side of (98.2) with respect to n_z and n_x gives the direction of the ray vector: $s_x/s_z = \varepsilon_\perp n_x/\varepsilon_\parallel n_z$. Thus the angle θ' between the ray vector and the optical axis and the angle θ satisfy the simple relation

$$\tan\theta' = (\varepsilon_\perp/\varepsilon_\parallel)\tan\theta. \tag{98.6}$$

The directions of $\mathbf{n}$ and $\mathbf{s}$ are the same only for waves propagated along or perpendicular to the optical axis.

The problem of the directions of polarization of the ordinary and extraordinary waves is very easily solved. It is sufficient to observe that the four vectors $\mathbf{E}$, $\mathbf{D}$, $\mathbf{s}$ and $\mathbf{n}$ are always coplanar. In the extraordinary wave $\mathbf{s}$ and $\mathbf{n}$ are not in the same direction, but lie in the same principal section. This wave is therefore polarized so that the vectors $\mathbf{E}$ and $\mathbf{D}$ lie in the same principal section as $\mathbf{s}$ and $\mathbf{n}$. The vectors $\mathbf{D}$ in the ordinary and extraordinary waves with the same direction of $\mathbf{n}$ (or $\mathbf{E}$, with the same direction of $\mathbf{s}$) are perpendicular. Hence the polarization of the ordinary wave is such that $\mathbf{E}$ and $\mathbf{D}$ lie in a plane perpendicular to the principal section.

An exception is formed by waves propagated in the direction of the optical axis. In this direction there is no difference between the ordinary and the extraordinary wave, and so their polarizations combine to give a wave which is, in general, elliptically polarized.

The refraction of a plane wave incident on the surface of a crystal is different from refraction at a boundary between two isotropic media. The laws of refraction and reflection are again obtained from the continuity of the component $\mathbf{n}_t$ of the wave vector which is tangential to the plane of separation. The wave vectors of the refracted and reflected waves therefore lie in the plane of incidence. In a crystal, however, two different refracted waves are formed, a phenomenon known as *double refraction* or *birefringence*. They correspond to the two possible values of the normal component n_n which satisfy Fresnel's equation for a given tangential component $\mathbf{n}_t$. It should also be remembered that the observed direction of propagation of the rays is determined not by the wave vector but by the ray vector $\mathbf{s}$, whose direction is different from that of $\mathbf{n}$ and in general does not lie in the plane of incidence.

In a uniaxial crystal, ordinary and extraordinary refracted waves are formed. The ordinary wave is entirely analogous to the refracted wave in isotropic bodies; in particular its ray vector (which is in the same direction as its wave vector) lies in the plane of incidence. The ray vector of the extraordinary wave in general does not lie in the plane of incidence.

PROBLEMS

Problem 1. Find the direction of the extraordinary ray when light incident from a vacuum is refracted at a surface of a uniaxial crystal which is perpendicular to its optical axis.

Solution. In this case the refracted ray lies in the plane of incidence, which we take as the xz-plane, with the z-axis normal to the surface. The x-component of the wave vector $n_x = \sin\vartheta$ (ϑ being the angle of incidence) is continuous; the component n_z for the refracted wave is found from (98.2):

$$n_z = \sqrt{\left(\varepsilon_\perp - \frac{\varepsilon_\perp}{\varepsilon_\parallel}\sin^2\vartheta\right)}.$$

The direction of the refracted ray is given by (98.6):

$$\tan\vartheta' = \frac{\varepsilon_\perp}{\varepsilon_\parallel}\frac{n_x}{n_z} = \frac{\sqrt{\varepsilon_\perp}\sin\vartheta}{\sqrt{[\varepsilon_\parallel(\varepsilon_\parallel - \sin^2\vartheta)]}},$$

where ϑ' is the angle of refraction.

Problem 2. Find the direction of the extraordinary ray when light is incident normally on a surface of a uniaxial crystal at any angle to the optical axis.

Solution. The refracted ray lies in the xz-plane, which passes through the normal to the surface (the z-axis) and the optical axis. Let α be the angle between these axes. The ray vector $\mathbf{s}$, whose components are proportional to the derivatives of the left-hand side of equation (98.2) with respect to the corresponding components of $\mathbf{n}$, is proportional to

$$\frac{\mathbf{n}}{\varepsilon_\parallel} + (\mathbf{n}\cdot\mathbf{l})\mathbf{l}\left(\frac{1}{\varepsilon_\perp} - \frac{1}{\varepsilon_\parallel}\right),$$

where $\mathbf{l}$ is a unit vector in the direction of the optical axis. In the present case the wave vector $\mathbf{n}$ is in the z-direction, so that

$$s_x \propto \cos\alpha\sin\alpha\left(\frac{1}{\varepsilon_\perp} - \frac{1}{\varepsilon_\parallel}\right), \qquad s_z \propto \frac{\sin^2\alpha}{\varepsilon_\parallel} + \frac{\cos^2\alpha}{\varepsilon_\perp}.$$

Hence we find

$$\tan\vartheta' = \frac{s_x}{s_z} = \frac{(\varepsilon_\parallel - \varepsilon_\perp)\sin 2\alpha}{\varepsilon_\parallel + \varepsilon_\perp + (\varepsilon_\parallel - \varepsilon_\perp)\cos 2\alpha}.$$

§99. Biaxial crystals

In biaxial crystals the three principal values of the tensor ε_{ik} are all different. The crystals of the triclinic, monoclinic and orthorhombic systems are of this type. In those of the triclinic system, the position of the principal dielectric axes is unrelated to any specific crystallographic direction; in particular, it varies with frequency, as do all the components ε_{ik}. In crystals of the monoclinic system, one of the principal dielectric axes is crystallographically fixed; it coincides with the twofold axis of symmetry, or is perpendicular to the plane of symmetry. The position of the other two principal axes depends on the frequency. Finally, in crystals of the orthorhombic system, the position of all three principal axes is fixed: they must coincide with the three mutually perpendicular twofold axes of symmetry.

The study of the optical properties of biaxial crystals involves the consideration of Fresnel's equation in its general form. We shall assume for definiteness that

$$\varepsilon^{(x)} < \varepsilon^{(y)} < \varepsilon^{(z)}. \tag{99.1}$$

To ascertain the form of the fourth-order surface defined by equation (97.10), let us begin by finding its intersections with the coordinate planes. Putting $n_z = 0$ in equation

(97.10), we find that the left-hand side is the product of two factors:

$$(n^2 - \varepsilon^{(z)})(\varepsilon^{(x)} n_x^2 + \varepsilon^{(y)} n_y^2 - \varepsilon^{(x)}\varepsilon^{(y)}) = 0.$$

Hence we see that the section by the xy-plane consists of the circle

$$n^2 = \varepsilon^{(z)} \qquad (99.2)$$

and the ellipse

$$\frac{n_x^2}{\varepsilon^{(y)}} + \frac{n_y^2}{\varepsilon^{(x)}} = 1, \qquad (99.3)$$

and by the assumption (99.1) the ellipse lies inside the circle. Similarly we find that the sections by the yz and xz planes are also composed of an ellipse and a circle; in the yz-plane the ellipse lies outside the circle, and in the xz-plane they intersect. Thus the wave-vector surface intersects itself, and is as shown in Fig. 54, where one octant is drawn.

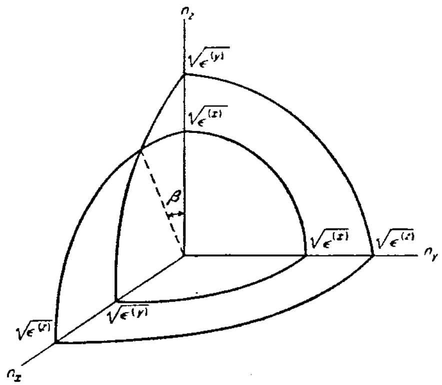

FIG. 54

This surface has four singular points of self-intersection, one in each quadrant of the xz-plane. The singular points of a surface whose equation is $f(n_x, n_y, n_z) = 0$ are given by the vanishing of all three first derivatives of the function f. Differentiating the left-hand side of (97.10), we obtain the equations

$$\left.\begin{aligned} n_x[\varepsilon^{(x)}(\varepsilon^{(y)}+\varepsilon^{(z)}) - \varepsilon^{(x)}n^2 - (\varepsilon^{(x)}n_x^2 + \varepsilon^{(y)}n_y^2 + \varepsilon^{(z)}n_z^2)] &= 0,\\ n_y[\varepsilon^{(y)}(\varepsilon^{(x)}+\varepsilon^{(z)}) - \varepsilon^{(y)}n^2 - (\varepsilon^{(x)}n_x^2 + \varepsilon^{(y)}n_y^2 + \varepsilon^{(z)}n_z^2)] &= 0,\\ n_z[\varepsilon^{(z)}(\varepsilon^{(x)}+\varepsilon^{(y)}) - \varepsilon^{(z)}n^2 - (\varepsilon^{(x)}n_x^2 + \varepsilon^{(y)}n_y^2 + \varepsilon^{(z)}n_z^2)] &= 0;\end{aligned}\right\} \qquad (99.4)$$

the equation (97.10) itself must, of course, be satisfied also. Since we know that the required directions of $\mathbf{n}$ lie in the xz-plane, we put $n_y = 0$, and the two remaining equations give

immediately†

$$n_x^2 = \frac{\varepsilon^{(z)}(\varepsilon^{(y)} - \varepsilon^{(x)})}{\varepsilon^{(z)} - \varepsilon^{(x)}}, \qquad n_z^2 = \frac{\varepsilon^{(x)}(\varepsilon^{(z)} - \varepsilon^{(y)})}{\varepsilon^{(z)} - \varepsilon^{(x)}}. \qquad (99.5)$$

The directions of these vectors $\mathbf{n}$ are inclined to the z-axis at an angle β such that

$$\frac{n_x}{n_z} = \pm\tan\beta = \pm\sqrt{\frac{\varepsilon^{(z)}(\varepsilon^{(y)} - \varepsilon^{(x)})}{\varepsilon^{(x)}(\varepsilon^{(z)} - \varepsilon^{(y)})}}. \qquad (99.6)$$

This formula determines lines in two directions in the xz-plane, each of which passes through two opposite singular points and is at an angle β to the z-axis. These lines are called the *optical axes* or *binormals* of the crystal; one of them is shown dashed in Fig. 54. The directions of the optical axes are evidently the only ones for which the wave vector has only one magnitude.‡

The properties of the ray surface are entirely similar. To derive the corresponding formulae, it is sufficient to replace $\mathbf{n}$ by $\mathbf{s}$ and ε by $1/\varepsilon$. In particular, there are two *optical ray axes* or *biradials*, also lying in the xz-plane and at an angle γ to the z-axis, where

$$\tan\gamma = \sqrt{\frac{\varepsilon^{(y)} - \varepsilon^{(x)}}{\varepsilon^{(z)} - \varepsilon^{(y)}}} = \sqrt{\frac{\varepsilon^{(x)}}{\varepsilon^{(z)}}}\tan\beta. \qquad (99.7)$$

Since $\varepsilon^{(x)} < \varepsilon^{(z)}$, $\gamma < \beta$.

The directions of corresponding vectors $\mathbf{n}$ and $\mathbf{s}$ are the same only for waves propagated along one of the coordinate axes (i.e. the principal dielectric axes). If $\mathbf{n}$ lies in one of the coordinate planes, $\mathbf{s}$ lies in that plane also. This rule, however, is subject to an important exception for wave vectors in the direction of the optical axes.

When the values of $\mathbf{n}$ given by (99.5) are substituted in the general formulae for $\mathbf{s}$ in terms of $\mathbf{n}$ (§97, Problem), these take the indeterminate form 0/0. The origin and meaning of this indeterminacy are quite evident from the following geometrical considerations. Near a singular point, the inner and outer parts of the wave-vector surface are cones with a common vertex. At the vertex, which is the singular point itself, the direction of the normal to the surface becomes indeterminate; and the direction of $\mathbf{s}$ as given by these formulae is just the direction of the normal. In fact the wave vector along the binormal corresponds to an infinity of ray vectors, whose directions occupy a certain conical surface, called the *cone of internal conical refraction*.§

To determine this cone of rays, we could investigate the directions of the normals near the singular point. It is more informative, however, to use a geometrical construction from the ray surface.

Fig. 55 (p. 344) shows one quadrant of the intersection of the ray surface with the xz-plane (continuous curves), and also the intersection of the wave-vector surface, on a different scale. The line OS is the biradial, and ON the binormal. Let $\mathbf{n}_N$ be the wave vector corresponding to the point N. It is easy to see that the singular point N on the wave-vector

† It is easy to see that the solution thus found is the only real solution of equations (99.4). If none of n_x, n_y, n_z is zero, the three equations (99.4) are inconsistent: they then involve only two unknowns, namely n^2 and $\varepsilon^{(x)}n_x^2 + \varepsilon^{(y)}n_y^2 + \varepsilon^{(z)}n_z^2$. If n_x or n_z is zero the solutions are imaginary.

‡ In the tensor ellipsoid (97.23) the binormals are the directions perpendicular to the circular sections of the ellipsoid. An ellipsoid has two such sections.

§ The phenomenon of conical refraction described below was predicted by W. R. Hamilton (1833).

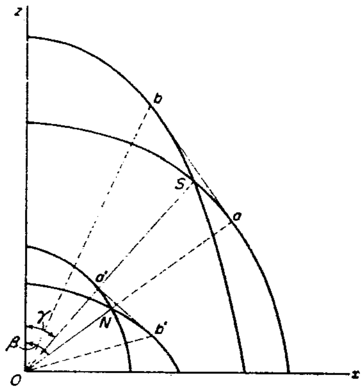

FIG. 55

surface corresponds to a singular tangent plane to the ray surface. This plane is perpendicular to ON, and touches the ray surface not at one point but along a curve, which is found to be a circle. In Fig. 55 the trace of this plane is shown by ab. This follows at once from the geometrical correspondence between the wave-vector surface and the ray surface (§97): if the tangent plane is drawn at any point $\mathbf{s}$ of the ray surface, then the perpendicular from the origin to this plane is in the same direction as the wave vector $\mathbf{n}$ corresponding to $\mathbf{s}$, and its length is $1/n$. In our case there must be an infinity of vectors $\mathbf{s}$ corresponding to the single value $\mathbf{n} = \mathbf{n}_N$; hence the points on the ray surface which represent these vectors $\mathbf{s}$ must lie in one tangent plane, which is perpendicular to $\mathbf{n}_N$. Thus in Fig. 55 the triangle Oab is the section of the cone of internal conical refraction by the xz-plane.

There is no especial difficulty in carrying out a quantitative calculation corresponding to this geometrical picture, but we shall not do so here, and give only the final formulae. The equations of the circle in which the cone of refraction cuts the ray surface are

$$\left.\begin{aligned}(\varepsilon^{(z)}-\varepsilon^{(x)})s_y{}^2+\{s_x\sqrt{[\varepsilon^{(x)}(\varepsilon^{(z)}-\varepsilon^{(y)})]}-s_z\sqrt{[\varepsilon^{(z)}(\varepsilon^{(y)}-\varepsilon^{(x)})]}\}\times&\\ \times\left(s_x\sqrt{\frac{\varepsilon^{(z)}-\varepsilon^{(y)}}{\varepsilon^{(x)}}}-s_z\sqrt{\frac{\varepsilon^{(y)}-\varepsilon^{(x)}}{\varepsilon^{(z)}}}\right)=0,&\\ s_x\sqrt{[\varepsilon^{(z)}(\varepsilon^{(y)}-\varepsilon^{(x)})]}+s_z\sqrt{[\varepsilon^{(x)}(\varepsilon^{(z)}-\varepsilon^{(y)})]}=\sqrt{[\varepsilon^{(z)}-\varepsilon^{(x)}]}.&\end{aligned}\right\}\quad(99.8)$$

The first of these equations is the equation of the cone of refraction if s_x, s_y, s_z are regarded as three independent variables. The second is the equation of the tangent plane to the ray surface. In particular, for $s_y = 0$ equation (99.8) gives the two equations

$$\frac{s_x}{s_z}=\sqrt{\frac{\varepsilon^{(z)}(\varepsilon^{(y)}-\varepsilon^{(x)})}{\varepsilon^{(x)}(\varepsilon^{(z)}-\varepsilon^{(y)})}},\qquad \frac{s_x}{s_z}=\sqrt{\frac{\varepsilon^{(x)}(\varepsilon^{(y)}-\varepsilon^{(x)})}{\varepsilon^{(z)}(\varepsilon^{(z)}-\varepsilon^{(y)})}},$$

which determine the directions of the extreme rays (respectively Oa and Ob in Fig. 55) in the section by the xz-plane. The former is along the binormal (cf. (99.6)), which is perpendicular to the tangent ab.

Similar results hold for the wave vectors corresponding to a given ray vector. The vector $\mathbf{s}$ along the biradial corresponds to an infinity of wave vectors, whose directions occupy the *cone of external conical refraction*. In Fig. 55 the triangle $Oa'b'$ is the section of this cone by the xz-plane. The corresponding formulae are again obtained by substituting $\mathbf{n}$ for $\mathbf{s}$ and $1/\varepsilon$ for ε in the formulae (99.8), and are

$$\left.\begin{aligned}\varepsilon^{(y)}(\varepsilon^{(z)}-\varepsilon^{(x)})n_y{}^2+[n_x\sqrt{(\varepsilon^{(z)}-\varepsilon^{(y)})}-n_z\sqrt{(\varepsilon^{(y)}-\varepsilon^{(x)})}]\times&\\ \times[n_x\varepsilon^{(x)}\sqrt{(\varepsilon^{(z)}-\varepsilon^{(y)})}-n_z\varepsilon^{(z)}\sqrt{(\varepsilon^{(y)}-\varepsilon^{(x)})}]=0,&\\ n_x\sqrt{(\varepsilon^{(y)}-\varepsilon^{(x)})}+n_z\sqrt{(\varepsilon^{(z)}-\varepsilon^{(y)})}=\sqrt{[\varepsilon^{(y)}(\varepsilon^{(z)}-\varepsilon^{(x)})]}.&\end{aligned}\right\}\quad(99.9)$$

In observations of the internal conical refraction† we can use a flat plate cut perpendicular to the binormal (Fig. 56). The surface of the plate is covered by a diaphragm of small aperture, which selects a narrow beam from a plane light wave (i.e. one whose wave vector is in a definite direction) incident on the plate. The wave vector in the wave transmitted into the plate is in the direction of the binormal, and so the rays are on the cone of internal refraction. The wave vector in the wave leaving the other side of the plate is the same as in the incident wave, and so the rays are on a circular cylinder.

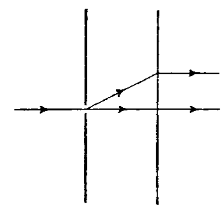

FIG. 56

To observe the external conical refraction, the plate must be cut perpendicular to the biradial, and both its surfaces must be covered by diaphragms having small apertures in exactly opposite positions. When the plate is illuminated by a convergent beam (i.e. one containing rays with all possible values of $\mathbf{n}$), the diaphragms admit to the plate rays with $\mathbf{s}$ along the biradial, and therefore with directions of $\mathbf{n}$ occupying the surface of the cone of external conical refraction. The light leaving the second aperture is therefore on a conical surface, although this does not exactly coincide with the cone of external refraction, on account of the refraction on leaving the plate.

The laws of refraction at the surface of a biaxial crystal for an arbitrary direction of incidence are extremely complex, and we shall not pause to discuss them here,‡ but only mention that, unlike what happens for a uniaxial crystal, both refracted waves are "extraordinary" and the rays of neither lie in the plane of incidence.

As specified in §97, we are describing the optics of transparent crystals, but we may note here that there is a property of biaxial crystals that may occur when absorption is taken into account.

† We shall describe only the principle of the experiment.

‡ A detailed account of the calculations may be found in the article by G. Szivessy, *Handbuch der Physik*, vol. XX, Chapter 11, Springer, Berlin, 1928.

Let us consider a homogeneous plane wave propagated in the crystal, in which **n** is a complex vector but its real and imaginary parts are in the same direction: $\mathbf{n} = n\mathbf{v}$, where $\mathbf{v}$ is a real unit vector, $n = n(\omega)$ a complex quantity. For a given $\mathbf{v}$, the dispersion equation (97.21) can be expanded as

$$n^{-4} - n^{-2}(\eta_{11}+\eta_{22}) + \eta_{11}\eta_{22} - \eta_{12}{}^2 = 0,$$

where $\eta_{ik} \equiv \varepsilon^{-1}{}_{ik}$, and 1 and 2 are tensor suffixes in the plane perpendicular to $\mathbf{v}$. This equation quadratic in n^{-2} has a multiple root if

$$\eta_{22} - \eta_{11} = \pm 2i\eta_{12}; \tag{99.10}$$

then $n^{-2} = \frac{1}{2}(\eta_{11}+\eta_{22})$. When absorption is present, the tensor $\eta_{ik} = \eta_{ik}' + \eta_{ik}''$ is complex.

In biaxial crystals, the tensor ellipsoids of η_{ik}' and η_{ik}'' have three unequal axes; the ratios of the axes are different for the two tensors (and so are their directions, in triclinic and monoclinic crystals). Under these conditions, the two-dimensional tensors $\eta_{\alpha\beta}'$ and $\eta_{\alpha\beta}''$ cannot in general be simultaneously brought to diagonal form. The angle ϑ between the principal axes of the two tensors is a function of two independent variables, the angles which specify the direction of $\mathbf{v}$. For a given frequency ω, therefore, there can exist a one-parameter set of directions of $\mathbf{v}$ for which $\vartheta = \frac{1}{4}\pi$. With this value of ϑ, the imaginary part of the complex equation (99.10) is satisfied identically; the real part is

$$\eta_2' - \eta_1' = \mp(\eta_2'' - \eta_1''), \tag{99.11}$$

where the suffixes 1 and 2 denote the principal values of the tensors concerned.† For any choice of the x_1 and x_2 axes, equations (97.21) now give

$$D_2/D_1 = (\eta_{22} - \eta_{11})/2\eta_{12} = \pm i,$$

the two signs on the right corresponding to those in (99.10). Thus the conditions $\vartheta = \frac{1}{4}\pi$ and (99.11) together determine, for each value of ω, a particular direction of $\mathbf{v}$ in which only a wave with circular polarization of one sign, left or right according to the sign for which (99.10) is satisfied, can be propagated (W. Voigt, 1902). This direction in the crystal is called the *singular optical axis* or the *circular optical axis*.

In accordance with the general theory of linear differential equations, the second independent solution of the field equations then contains, not only the exponential factor $e^{in\mathbf{v}\cdot\mathbf{r}}$ (which includes the damping), but also the factor $a + b\mathbf{v}\cdot\mathbf{r}$ linear in the coordinates.‡ The polarization of this wave varies along the ray, but ultimately, as $\mathbf{v}\cdot\mathbf{r}$ increases, a circular polarization is established similar to that in the first wave, as is obvious if we note that in the limit concerned the substitution of the solution in the field equations involves differentiating only the exponential factor, and the difference between the two solutions then disappears.

We should again emphasize the difference between the singular axis and the case where the dispersion equation necessarily has a double root because of the symmetry of the crystal. For light propagated along the optical axis of a uniaxial crystal, the tensor $\eta_{\alpha\beta}$ has

† This is easily proved by taking the x_1 and x_2 axes along the principal axes of the tensor $\eta_{\alpha\beta}'$ and expressing the components of $\eta_{\alpha\beta}''$ in terms of its principal values.

‡ This solution has to be taken into account, for example, in problems of the reflection and refraction of light propagated along the singular axis.

the form $\eta\delta_{\alpha\beta}$, and the condition (99.10) is satisfied identically. Equations (97.21) then allow two independent solutions with different polarizations.

§100. Double refraction in an electric field

An isotropic body becomes optically anisotropic when placed in a static electric field. This anisotropy may be regarded as the result of a change in the permittivity due to the static field. Although this change is relatively very slight, it is important here because it leads to a qualitative change in the optical properties of bodies.

In this section we denote by **E** the static electric field in the body,† and expand the dielectric tensor ε_{ik} in powers of **E**. In an isotropic body in the zero-order approximation, we have $\varepsilon_{ik} = \varepsilon^{(0)}\delta_{ik}$. There can be no terms in ε_{ik} which are of the first order in the field, since in an isotropic body there is no constant vector with which a tensor of rank two linear in **E** could be constructed. The next terms in the expansion of ε_{ik} must therefore be quadratic in the field. From the components of the vector **E** we can form two symmetrical tensors of rank two, $E^2\delta_{ik}$ and E_iE_k. The former does not alter the symmetry of the tensor $\varepsilon^{(0)}\delta_{ik}$, and the addition of it amounts to a small correction in the scalar constant $\varepsilon^{(0)}$, which evidently does not result in optical anisotropy and is therefore of no interest. Thus we arrive at the following form of the dielectric tensor as a function of the field:

$$\varepsilon_{ik} = \varepsilon^{(0)}\delta_{ik} + \alpha E_iE_k, \tag{100.1}$$

where α is a scalar constant.

One of the principal axes of this tensor coincides with the direction of the electric field, and the corresponding principal value is

$$\varepsilon_{\parallel} = \varepsilon^{(0)} + \alpha E^2. \tag{100.2}$$

The other two principal values are both equal to

$$\varepsilon_{\perp} = \varepsilon^{(0)}, \tag{100.3}$$

and the position of the corresponding principal axes in a plane perpendicular to the field is arbitrary. Thus an isotropic body in an electric field behaves optically as a uniaxial crystal (the *Kerr effect*).

The change in optical symmetry in an electric field may occur in a crystal also (for example, an optically uniaxial crystal may become biaxial, and a cubic crystal may cease to be optically isotropic), and here the effect may be of the first order in the field. This linear effect corresponds to a dielectric tensor of the form

$$\varepsilon_{ik} = \varepsilon_{ik}{}^{(0)} + \alpha_{ikl}E_l, \tag{100.4}$$

where the coefficients α_{ikl} form a tensor of rank three symmetrical in the suffixes i and k. The symmetry of this tensor is the same as that of the piezoelectric tensor. The effect in question therefore occurs in the twenty crystal classes which admit piezoelectricity.

§101. Magnetic–optical effects

In the presence of a static magnetic field **H**,‡ the tensor $\varepsilon_{ik}(\omega; \mathbf{H})$ is no longer

† Not to be confused with the weak variable electric field of the wave.

‡ Not to be confused with the weak variable field of the electromagnetic wave.

symmetrical. The generalized principle of symmetry of the kinetic coefficients relates the components ε_{ik} and ε_{ki} in different fields:

$$\varepsilon_{ik}(\mathbf{H}) = \varepsilon_{ki}(-\mathbf{H}). \tag{101.1}$$

The condition that absorption be absent requires that the tensor should be Hermitian:

$$\varepsilon_{ik} = \varepsilon_{ki}^*, \tag{101.2}$$

as is seen from (96.5), but not that it should be real. Equation (101.2) implies only that the real and imaginary parts of ε_{ik} must be respectively symmetrical and antisymmetrical:

$$\varepsilon_{ik}' = \varepsilon_{ki}', \qquad \varepsilon_{ik}'' = -\varepsilon_{ki}''. \tag{101.3}$$

Using (101.1), we have

$$\left.\begin{aligned} \varepsilon_{ik}'(\mathbf{H}) &= \varepsilon_{ki}'(\mathbf{H}) = \varepsilon_{ik}'(-\mathbf{H}), \\ \varepsilon_{ik}''(\mathbf{H}) &= -\varepsilon_{ki}''(\mathbf{H}) = -\varepsilon_{ik}''(-\mathbf{H}), \end{aligned}\right\} \tag{101.4}$$

i.e. in a non-absorbing medium ε_{ik}' is an even function of $\mathbf{H}$, and ε_{ik}'' an odd function.

The inverse tensor $\varepsilon^{-1}{}_{ik}$ evidently has the same symmetry properties, and is more convenient for use in the following calculations. To simplify the notation we shall write†

$$\varepsilon^{-1}{}_{ik} = \eta_{ik} = \eta_{ik}' + i\eta_{ik}'', \tag{101.5}$$

as already used above.

Any antisymmetrical tensor of rank two is equivalent (dual) to some axial vector; let the vector corresponding to the tensor η_{ik}'' be $\mathbf{G}$. Using the antisymmetrical unit tensor e_{ikl}, we can write the relation between the components η_{ik}'' and G_i as

$$\eta_{ik}'' = e_{ikl} G_l, \tag{101.6}$$

or, in components, $\eta_{xy}'' = G_z, \eta_{zx}'' = G_y, \eta_{yz}'' = G_x$. The relation $E_i = \eta_{ik} D_k$ between the electric field and induction becomes

$$E_i = (\eta_{ik}' + ie_{ikl} G_l) D_k = \eta_{ik}' D_k + i(\mathbf{D}\times\mathbf{G})_i. \tag{101.7}$$

There is a similar linear relation

$$D_i = \varepsilon_{ik}' E_k + i(\mathbf{E}\times\mathbf{g})_i. \tag{101.8}$$

The connection between the coefficients in (101.7) and (101.8) is given by

$$\eta_{ik}' = \frac{1}{|\varepsilon|}\left\{\varepsilon'^{-1}{}_{ik}|\varepsilon'| - g_i g_k\right\}, \qquad G_i = -\frac{1}{|\varepsilon|}\varepsilon_{ik}' g_k, \tag{101.9}$$

where $|\varepsilon|$ and $|\varepsilon'|$ are the determinants of the tensors ε_{ik} and ε_{ik}'; cf. §22, Problem. A medium in which the relation between $\mathbf{E}$ and $\mathbf{D}$ is of this form is said to be *gyrotropic*. The vector $\mathbf{g}$ is called the *gyration vector*, and $\mathbf{G}$ the *optical activity vector*.

We shall give a general discussion of the nature of waves propagated in an arbitrary gyrotropic medium, assumed anisotropic, with no restriction on the magnitude of the magnetic field.†

† Of course, η_{ik}' and η_{ik}'' are not the tensors inverse to ε_{ik}' and ε_{ik}''.

We take the direction of the wave vector as the z-axis. Then equations (97.21) become

$$\left(\eta_{\alpha\beta} - \frac{1}{n^2}\delta_{\alpha\beta}\right) D_\beta = \left(\eta_{\alpha\beta}' + i\eta_{\alpha\beta}'' - \frac{1}{n^2}\delta_{\alpha\beta}\right) D_\beta = 0, \tag{101.10}$$

where the suffixes α, β take the values x, y. The directions of the x and y axes are taken along the principal axes of the two-dimensional tensor $\eta_{\alpha\beta}'$; and we denote the corresponding principal values of this tensor by $1/n_{01}^2$ and $1/n_{02}^2$. Then the equations become

$$\left.\begin{aligned} \left(\frac{1}{n_{01}^2} - \frac{1}{n^2}\right) D_x + iG_z D_y &= 0, \\ -iG_z D_x + \left(\frac{1}{n_{02}^2} - \frac{1}{n^2}\right) D_y &= 0. \end{aligned}\right\} \tag{101.11}$$

The condition that the determinant of these equations vanishes gives an equation quadratic in n^2:

$$\left(\frac{1}{n^2} - \frac{1}{n_{01}^2}\right)\left(\frac{1}{n^2} - \frac{1}{n_{02}^2}\right) = G_z^2, \tag{101.12}$$

whose roots give the two values of n for a given direction of $\mathbf{n}$:‡

$$\frac{1}{n^2} = \frac{1}{2}\left(\frac{1}{n_{01}^2} + \frac{1}{n_{02}^2}\right) \pm \sqrt{\left[\frac{1}{4}\left(\frac{1}{n_{01}^2} - \frac{1}{n_{02}^2}\right)^2 + G_z^2\right]}. \tag{101.13}$$

Substituting these values in equations (101.11), we find the corresponding ratios D_y/D_x:

$$\frac{D_y}{D_x} = \frac{i}{G_z}\left\{\frac{1}{2}\left(\frac{1}{n_{01}^2} - \frac{1}{n_{02}^2}\right) \mp \sqrt{\left[\frac{1}{4}\left(\frac{1}{n_{01}^2} - \frac{1}{n_{02}^2}\right)^2 + G_z^2\right]}\right\}. \tag{101.14}$$

The purely imaginary value of the ratio D_y/D_x signifies that the waves are elliptically polarized, and the principal axes of the ellipses are the x and y axes. The product of the two values of the ratio is easily seen to be unity. Thus, if in one wave $D_y = i\rho D_x$, where the real quantity ρ is the ratio of the axes of the polarization ellipse, then in the other wave $D_y = -iD_x/\rho$. This means that the polarization ellipses of the two waves have the same axis ratio, but are rotated 90° relative to each other, and the directions of rotation in them are opposite (Fig. 57, p. 350).

If the vectors $\mathbf{D}$ in the two waves are denoted by $\mathbf{D}_1$ and $\mathbf{D}_2$, these relations may be written $\mathbf{D}_1\cdot\mathbf{D}_2^* = D_{1x}D_{2x}^* + D_{1y}D_{2y}^* = 0$. This is a general property of the eigenvectors on reduction to diagonal form of an Hermitian tensor (in this case, the tensor $\eta_{\alpha\beta}$).

† The medium is again assumed non-magnetic with respect to the variable field of the electromagnetic wave, i.e. $\mu_{ik}(\omega) = \delta_{ik}$. This, however, does not exclude a static field magnetizing the medium (i.e. the static permeability may differ from unity).

The properties derived for $\varepsilon_{ik}(\omega)$ are equally applicable to the tensor $\mu_{ik}(\omega)$ in a frequency range where the dispersion of the magnetic permeability is of importance.

‡ When there is no field, $\mathbf{G} = 0$ and $n = n_{01}$ or n_{02}. It should be remembered, however, that when the field is present n_{01} and n_{02} in equation (101.12) are not in general the values of n for $\mathbf{H} = 0$, since not only $\mathbf{G}$ but also the components η_{ik}' depend on the field.

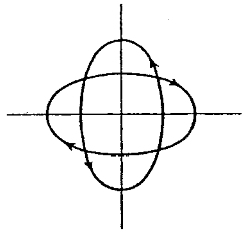

FIG. 57

The components G_i and η_{ik}' are functions of the magnetic field. If, as usually happens, the magnetic field is fairly weak, we can expand in powers of the field. The vector **G** is zero in the absence of the field, and so for a weak field we can put

$$G_i = f_{ik} H_k, \tag{101.15}$$

where f_{ik} is a tensor of rank two, in general not symmetrical. This dependence is in accordance with the general rule whereby, in a transparent medium, the components of the antisymmetrical tensor η_{ik}'' (and ε_{ik}'') must be odd functions of **H**. The symmetrical tensor components η_{ik}' are even functions of the magnetic field. The first correction terms (which do not appear in the absence of the field) in η_{ik}' are therefore quadratic in the field. When second-order quantities are neglected, formulae (101.9) reduce to the simpler form

$$\eta_{ik}' = \varepsilon'^{-1}{}_{ik}.$$

In the general case of an arbitrarily directed wave vector, the magnetic field has little effect on the propagation of light in the crystal, causing only a slight ellipticity of the oscillations, with an axis ratio of the polarization ellipse which is small (of the first order with respect to the field).

The directions of the optical axes (and neighbouring directions) form an exception. The two values of n are equal in the absence of the field. The roots of equation (101.12) then differ from these values by first-order quantities,† and the resulting effects are analogous to those in isotropic bodies, which we shall now consider.

The magnetic–optical effect in isotropic bodies (and in crystals of the cubic system) is of particular interest on account of its nature and its comparatively large magnitude.

Neglecting second-order quantities, we have $\eta_{ik}' = \varepsilon^{-1}\delta_{ik}$, where ε is the permittivity of the isotropic medium in the absence of the magnetic field. The relation between **D** and **E** is

$$\mathbf{E} = \frac{1}{\varepsilon}\mathbf{D} + i\mathbf{D}\times\mathbf{G}, \quad \mathbf{D} = \varepsilon\mathbf{E} + i\mathbf{E}\times\mathbf{g}; \tag{101.16}$$

in the same approximation, the vectors **g** and **G** are related by

$$\mathbf{G} = -\mathbf{g}/\varepsilon^2. \tag{101.17}$$

† It should be noticed that the two roots of (101.12) do not become equal. The geometrical significance of this is that the two parts of the wave-vector surface are separated.

The dependence of **g** (or **G**) on the external field reduces in an isotropic medium to simple proportionality:

$$\mathbf{g} = f\mathbf{H}, \tag{101.18}$$

in which the scalar constant f may be either positive or negative.

In equation (101.12) we now have $n_{01} = n_{02} \equiv n_0 = \sqrt{\varepsilon}$, the refractive index in the absence of the field. Hence $1/n^2 = \mp G_z + 1/n_0^2$ or, to the same accuracy,

$$n_{\mp}^2 = n_0^2 \pm n_0^4 G_z = n_0^2 \mp g_z. \tag{101.19}$$

Since the z-axis is in the direction of **n**, we can write this formula, to the same accuracy, in the vector form

$$\left(\mathbf{n} \pm \frac{1}{2n_0}\mathbf{g}\right)^2 = n_0^2. \tag{101.20}$$

Hence we see that the wave-vector surface in this case consists of two spheres with radius n_0, whose centres are at distances $\pm g/2n_0$ from the origin in the direction of **G**.

A different polarization of the wave corresponds to each of the two values of n: we have

$$D_x = \mp i D_y, \tag{101.21}$$

where the signs correspond to those in (101.19). The equality of the magnitudes of D_x and D_y, and their phase difference of $\mp\frac{1}{2}\pi$, signify a circular polarization of the wave, with the direction of rotation of the vector **D** respectively anticlockwise and clockwise looking along the wave vector (or, to use the customary expressions, with *right-hand* and *left-hand* polarization respectively).

The difference between the refractive indices in the left-hand and right-hand polarized waves has the result that two circularly polarized refracted waves are formed at the surface of a gyrotropic body. This phenomenon is called *double circular refraction*.

Let a linearly polarized plane wave be incident normally on a slab of thickness l. We take the direction of incidence as the z-axis, and that of the vector **E** (= **D**) in the incident wave as the x-axis. The linear oscillation can be represented as the sum of two circular oscillations with opposite directions of rotation, which are then propagated through the slab with different wave numbers $k_{\pm} = \omega n_{\pm}/c$. Arbitrarily taking the wave amplitude as unity, we have $D_x = \frac{1}{2}[\exp(ik_+ z) + \exp(ik_- z)]$, $D_y = \frac{1}{2}i[-\exp(ik_+ z) + \exp(ik_- z)]$, or, putting $k = \frac{1}{2}(k_+ + k_-)$ and $\kappa = \frac{1}{2}(k_+ - k_-)$,

$$D_x = \tfrac{1}{2}e^{ikz}(e^{i\kappa z} + e^{-i\kappa z}) = e^{ikz}\cos\kappa z,$$
$$D_y = \tfrac{1}{2}ie^{ikz}(-e^{i\kappa z} + e^{-i\kappa z}) = e^{ikz}\sin\kappa z.$$

When the wave leaves the slab we have

$$D_y/D_x = \tan\kappa l = \tan(l\omega g/2cn_0). \tag{101.22}$$

Since this ratio is real, the wave remains linearly polarized, but the direction of polarization is changed (the *Faraday effect*). The angle through which the plane of polarization is rotated is proportional to the path traversed by the wave; the angle per unit length in the direction of the wave vector is

$$(\omega g/2cn_0)\cos\theta, \tag{101.23}$$

where θ is the angle between **n** and **g**.

It should be noticed that, when the direction of the magnetic field is given, the direction of rotation of the plane of polarization (with respect to the direction of **n**) is reversed (left-hand becoming right-hand, and vice versa) when the sign of **n** is changed. If the ray traverses the same path twice in opposite directions, the total rotation of the plane of polarization is therefore double the value resulting from a single traversal.

For $\theta = \frac{1}{2}\pi$ (the wave vector perpendicular to the magnetic field), the effect linear in the field given by formulae (101.19) disappears, in accordance with the general rule stated above that only the component of **g** in the direction of **n** affects the propagation of light. For angles θ close to $\frac{1}{2}\pi$ we must therefore take account of the terms proportional to the square of the field, and in particular these terms must be included in the tensor η_{ik}'. By virtue of the axial symmetry about the direction of the field, two principal values of the symmetrical tensor η_{ik}' are equal, as for a uniaxial crystal. We shall take the x-axis in the direction of the field, and denote by $\eta_\parallel$ and $\eta_\perp$ the principal values of η_{ik}' in the directions parallel and perpendicular to the magnetic field. The difference $\eta_\parallel - \eta_\perp$ is proportional to H^2.

Let us consider the purely quadratic effect (called the *Cotton–Mouton effect*) which occurs when **n** and **g** are perpendicular. In equations (101.11) and (101.12) we have $G_z = 0$, and $1/n_{01}^2$, $1/n_{02}^2$ are respectively $\eta_\parallel, \eta_\perp$. Thus in one wave we have $1/n^2 = \eta_\parallel$, $D_y = 0$; this wave is linearly polarized, and the vector **D** is parallel to the x-axis. In the other wave $1/n^2 = \eta_\perp$, $D_x = 0$, i.e. **D** is parallel to the y-axis. Let linearly polarized light be incident normally on a slab in a magnetic field parallel to its surface. The two components in the slab (with vectors **D** in the xz and yz planes) are propagated with different values of n. Consequently the light leaving the slab is elliptically polarized.

Lastly, let us consider one other peculiar effect that occurs in a medium whose optical activity vector (101.15) is linear in the (static) magnetic field, namely the magnetization of a non-magnetic transparent medium by a variable electric field (L. P. Pitaevskiĭ, 1960).

We start from the general formula (31.6)

$$-\mathbf{B}/4\pi = \partial \tilde{U}/\partial \mathbf{H},$$

and take account of the contribution to $\tilde{U}$ from the variable electric field, which is given by (80.11). According to the theorem of small increments to thermodynamic quantities, the change $\delta\tilde{U}$ in this contribution when the permittivity changes by a small amount is (expressed in terms of the appropriate variables) the same as the change $\delta\tilde{F}$ in the free energy. For the latter we can use formula (14.1), with an obvious generalization to anisotropic media; the fact that this formula remains valid for a variable field (not a static field as in §14) in a dispersive transparent medium has been mentioned in §81.† We thus have

$$\begin{aligned}\delta\tilde{U} &= -\delta\varepsilon_{ik} E_i E_k^*/16\pi \\ &= \delta\eta_{ik} D_i D_k^*/16\pi; \qquad (101.24)\end{aligned}$$

the extra factor $\frac{1}{2}$ takes account of the complex representation of **E**. The second equation (101.24) follows because the definition $\varepsilon_{il}\eta_{lk} = \delta_{ik}$ gives $\varepsilon_{il}\,\delta\eta_{lk} = -\eta_{lk}\,\delta\varepsilon_{il}$.‡

† The tilde above the symbol U refers here to the magnetic variables, not the electric ones. To simplify the notation, we omit the sign of time averaging from $\tilde{U}$.

‡ To derive (101.24) directly, it would be necessary to consider a dielectric-filled resonator instead of the oscillatory circuit discussed in §81. By calculating the change in frequency due to a small change in the permittivity (cf. §90, Problem 4) and using the adiabatic invariant theorem, we find the change in the resonator energy.

Now regarding the permittivity variation as the result of changing the static magnetic field, we write

$$-\frac{\mathbf{B}}{4\pi} = \frac{\partial \tilde{U}_0}{\partial \mathbf{H}} + \frac{\partial \eta_{ik}}{\partial \mathbf{H}}\frac{D_i D_k^*}{16\pi},$$

where $\tilde{U}_0$ refers to the medium in the absence of the electric field. If the medium itself is non-magnetic ($\mu = 1$), then $\partial\tilde{U}_0/\partial\mathbf{H} = -\mathbf{H}/4\pi$. The magnetization $\mathbf{M} = (\mathbf{B}-\mathbf{H})/4\pi$ is then

$$\mathbf{M} = -\frac{\partial \eta_{ik}}{\partial \mathbf{H}}\frac{D_i D_k^*}{16\pi}.$$

In the absence of the external magnetic field, the derivative $\partial\eta_{ik}/\partial\mathbf{H}$ is to be taken at $\mathbf{H} = 0$. With η_{ik} from (101.6), and (101.15), we obtain finally the following expression for the magnetization due to the variable electric field:

$$M_l = -(i/16\pi)e_{ikm} f_{ml} D_i D_k^*; \qquad (101.25)$$

this is quadratic in the electric field. If the medium is isotropic in the absence of the magnetic field, then $f_{ml} = f\delta_{ml}$ and

$$\mathbf{M} = -if\,\mathbf{D}\times\mathbf{D}^*/16\pi. \qquad (101.26)$$

For a linearly polarized field, the vector **D** can differ from a real quantity only by a phase factor; then **D** and **D*** are collinear, and (101.25) or (101.26) is zero. There is thus a magnetization only in the presence of a rotating electric field. This effect is in a sense the opposite of the rotation of the polarization plane in a magnetic field, and is expressed in terms of the same tensor f_{ik}; it is therefore called the *inverse Faraday effect*.

PROBLEMS

PROBLEM 1. Show by direct calculation that the direction of the (time) averaged Poynting vector in a wave propagated in a transparent gyrotropic medium is the same as that of the group velocity.

SOLUTION. According to (59.9a),

$$\bar{\mathbf{S}} = c\,\mathrm{re}\,\mathbf{E}^*\times\mathbf{H}/8\pi,$$

E and **H** being expressed in complex form. Proceeding as in the derivation of (97.16), we multiply equations (97.15) by **E*** and **H*** respectively:

$$\mathbf{E}^*\cdot\delta\mathbf{D} = \mathbf{H}^*\cdot\delta\mathbf{H} + (\mathbf{E}^*\times\mathbf{H})\cdot\delta\mathbf{n},$$

$$\mathbf{H}^*\cdot\delta\mathbf{H} = \mathbf{D}^*\cdot\delta\mathbf{E} + (\mathbf{E}\times\mathbf{H}^*)\cdot\delta\mathbf{n}.$$

Adding these and noting that $\mathbf{E}^*\cdot\delta\mathbf{D} = \mathbf{D}^*\cdot\delta\mathbf{E}$, since the tensor ε_{ik} is Hermitian, we find the required result:

$$\delta\mathbf{n}\cdot\mathrm{re}\,(\mathbf{E}^*\times\mathbf{H}) = 0.$$

PROBLEM 2. Determine the directions of the rays when a ray incident from a vacuum is refracted at the surface of an isotropic body in a magnetic field.

SOLUTION. The direction of the ray vector **s** is given by the normal to the wave-vector surface. Differentiating the left-hand side of equation (101.20) with respect to the components of the vector **n**, we find that **s** is proportional to $\mathbf{n}\pm\mathbf{g}/2n_0$. The square of the latter expression is n_0^2, and so the unit vector in the direction of the ray is given by

$$\frac{\mathbf{s}}{s} = \frac{1}{n_0}\left(\mathbf{n}\pm\frac{1}{2n_0}\mathbf{g}\right). \qquad (1)$$

Let the angle of incidence be θ. The refracted rays do not in general lie in the plane of incidence, and their directions are given by the angle θ' to the normal to the surface and the azimuth ϕ' measured from the plane of incidence. We take the latter as the xz-plane, with the z-axis perpendicular to the surface. The components n_x and n_y of the wave vector are unaltered by refraction. In the incident ray they are $n_x = \sin\theta$, $n_y = 0$. Substituting these values in (1), we find the x and y components of the unit vector $\mathbf{s}/s$, which give immediately the directions of the refracted rays:

$$\sin\theta'\cos\phi' = \frac{1}{n_0}\sin\theta \pm \frac{1}{2n_0^2}g_x,$$

$$\sin\theta'\sin\phi' = \pm\frac{1}{2n_0^2}g_y.$$

When the angle of incidence is not small, the azimuth ϕ' is small, and we can write

$$\phi' = \pm g_y/2n_0\sin\theta,$$

$$\sin\theta' = \frac{\sin\theta}{n_0} \pm \frac{g_x}{2n_0^2}.$$

For normal incidence ($\theta = 0$) we take the xz-plane through the vector $\mathbf{g}$; then $\phi' = 0$, and $\theta' \cong \sin\theta' = \pm g_x/2n_0^2$. Although this formula does not involve g_z, it is not valid if $g_z = 0$, since the approximation linear in the field is inadequate when $\mathbf{n}$ and $\mathbf{g}$ are perpendicular.

PROBLEM 3. Determine the polarization of the reflected light when a linearly polarized wave is incident normally from a vacuum on the surface of a body rendered anisotropic by a magnetic field.

SOLUTION. For normal incidence the direction of the wave vector is unaltered by the passage of the wave into the medium. In all three waves (incident, refracted and reflected) the vectors $\mathbf{H}$ are therefore parallel to the surface (the xy-plane). The electric vector $\mathbf{E}$ in the incident and reflected waves is also parallel to the xy-plane; in the refracted wave $E_z \neq 0$, but the relation between the x and y components of $\mathbf{E}$ and $\mathbf{H}$ is the same as in an isotropic body ($H_x = -nE_y$, $H_y = nE_x$). If the polarization of the incident wave is the same as that of one of the two types of wave which can be propagated in the anisotropic (refracting) medium concerned, with the given direction of $\mathbf{n}$, then there is only one refracted wave, which has this polarization. The problem is then formally identical with that of reflection from an isotropic body, and the fields $\mathbf{E}_1$ and $\mathbf{E}_0$ in the reflected and incident waves are related by

$$\mathbf{E}_1 = (1-n)\,\mathbf{E}_0/(1+n), \tag{1}$$

where n is the refractive index corresponding to this polarization.

The linear polarization can be regarded as resulting from the superposition of two circular polarizations with opposite directions of rotation. If $\mathbf{E}_0$ in the incident wave is in the x-direction, we put $\mathbf{E}_0 = \mathbf{E}_0^+ + \mathbf{E}_0^-$, where $E_0{}^+{}_x = iE_0{}^+{}_y = \frac{1}{2}E_0$, $E_0{}^-{}_x = -iE_0{}^-{}_y = \frac{1}{2}E_0$. Using formula (1) for each wave, with $n_\pm$ given by (101.19), we obtain

$$E_{1x} = \tfrac{1}{2}E_0\left[\frac{1-n_+}{1+n_+} + \frac{1-n_-}{1+n_-}\right] \cong E_0\frac{1-n_0}{1+n_0},$$

$$E_{1y} = \tfrac{1}{2}iE_0\left[\frac{1-n_-}{1+n_-} - \frac{1-n_+}{1+n_+}\right] \cong iE_0\frac{g\cos\theta}{n_0(1+n_0)^2},$$

where θ is the angle between the direction of incidence and the vector $\mathbf{g}$. Hence we see that the reflected wave is elliptically polarized, the major axis of the ellipse being in the x-direction, and the ratio of the minor and major axes being $(g\cos\theta)/n_0(n_0^2-1)$.

PROBLEM 4. Determine the limiting form of the frequency dependence of the gyration vector at high frequencies.

SOLUTION. The calculations are similar to those in §78, except that the electron equation of motion must include the Lorentz force due to the static external magnetic field $\mathbf{H}$:

$$m\frac{d\mathbf{v}'}{dt} = e\mathbf{E}_0 e^{-i\omega t} + e\mathbf{v}'\times\mathbf{H}/c,$$

where $e = -|e|$ is the electron charge. If $\omega \gg |e|H/mc$, this equation can be solved by successive approximations. As far as terms of the first order in $\mathbf{H}$ we have

$$\mathbf{v}' = \frac{ie}{m\omega}\mathbf{E} - \frac{e^2}{m^2\omega^2 c}\mathbf{E}\times\mathbf{H},$$

and the induction is then

$$\mathbf{D} = \varepsilon(\omega)\,\mathbf{E} + if(\omega)\,\mathbf{E}\times\mathbf{H},$$

where $\varepsilon(\omega)$ is given by (78.1) and $f(\omega) = -4\pi Ne^3/cm^2\omega^3 = (|e|/2mc)\,d\varepsilon/d\omega$ (H. Becquerel, 1897).

§102. Mechanical–optical effects

Besides the electric–optical and magnetic–optical effects, there are other ways in which the optical symmetry of a medium can be changed by external agencies. These include, first of all, the effect of elastic deformations on the optical properties of solids. In particular, such deformations may render an isotropic solid body optically anisotropic. Such phenomena are described by the inclusion in $\varepsilon_{ik}(\omega)$ of additional terms proportional to the components of the strain tensor. The corresponding formulae are exactly the same as (16.1) and (16.6) for the static permittivity, except that the coefficients are now functions of frequency. In the deformation of an isotropic body, for example, we have

$$\varepsilon_{ik} = \varepsilon^{(0)}\delta_{ik} + a_1 u_{ik} + a_2 u_{ll}\delta_{ik}. \tag{102.1}$$

The coefficients $a_1(\omega)$ and $a_2(\omega)$ are called *elastic–optical constants.*

Another case is the occurrence of optical anisotropy in a non-uniformly moving fluid. The corresponding general expression for the dielectric tensor is

$$\varepsilon_{ik} = \varepsilon^{(0)}{}_{ik} + \lambda_1\left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k}\right) + \tfrac{1}{2}i\lambda_2\left(\frac{\partial v_k}{\partial x_i} - \frac{\partial v_i}{\partial x_k}\right), \tag{102.2}$$

and represents the first terms in an expansion of ε_{ik} in powers of the derivatives of the velocity. The condition that absorption be absent (ε_{ik} is Hermitian) means that $\lambda_1(\omega)$ and $\lambda_2(\omega)$ must be real; $\varepsilon^{(0)}(\omega)$ is the permittivity of the fluid at rest. In an incompressible fluid $\partial v_l/\partial x_l \equiv \operatorname{div}\mathbf{v} = 0$, and the last two terms in (102.2) give zero on contraction.

To investigate the electromagnetic properties of the moving fluid, we have to combine the formulae (76.9)–(76.11) for the electrodynamics of moving dielectrics (with a velocity $\mathbf{v}$ that depends on the coordinates) with (102.2). Here, however, the terms which contain both the velocity and its derivatives are to be neglected, as being outside the accuracy of the formulae.

The second and third terms in (102.2) are respectively symmetrical and antisymmetrical in the suffixes i,k. For uniform rotation of the fluid we have $\mathbf{v} = \mathbf{\Omega}\times\mathbf{r}$, where $\mathbf{\Omega}$ is the angular velocity of rotation, and the symmetrical term is zero. The antisymmetrical term is $i\lambda_2 e_{ikl}\Omega_l$, so that the medium becomes gyrotropic, with gyration vector

$$\mathbf{g} = \lambda_2\mathbf{\Omega}. \tag{102.3}$$

The quantity λ_2 contains contributions from two effects: the dispersion of the permittivity, and the influence of Coriolis forces on it.

In a frame of reference moving with a given element of the fluid, the amplitude $\mathbf{E}_0$ of a monochromatic wave (in the laboratory frame) rotates with angular velocity $-\mathbf{\Omega}$, i.e. becomes a function of time satisfying the equation

$$\partial\mathbf{E}_0/\partial t = -\mathbf{\Omega}\times\mathbf{E}_0.$$

In this sense the wave becomes quasi-monochromatic, and the relation between $\mathbf{D}$ and $\mathbf{E}$ is