Homework solutions 9

1.)

$$\rho(r') = \delta(x')\,\delta(y')\,\lambda(z')$$

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}}\,dx'dy'dz'$$

$a \to z'$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 + \frac{z'}{r}\cos\theta + \left(\frac{z}{r}\right)^2\left\{\frac{3\cos^2\theta - 1}{2}\right\} + \ldots\right]$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{z}{r}\right)^{\ell} P_{\ell}(\cos\theta) \qquad r > a$$

Legendre polynomials

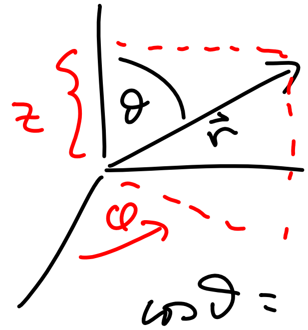
$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\delta(x')\delta(y')\lambda(z')}{r}\sum_{\ell=0}^{\infty}\left(\frac{z'}{r}\right)^{\ell} P_{\ell}(\cos\theta)\,dx'dy'dz'$$

$$\int_{-\infty}^{\infty}\delta(x')\,dx' = 1 \qquad \int_{-\infty}^{\infty}\delta(y')\,dy' = 1$$

$$V = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty} M_{\ell}\,\frac{P_{\ell}(\cos\theta)}{r^{\ell+1}}\;;\quad M_{\ell} = \int \lambda(z')(z')^{\ell}\,dz'$$

2.)

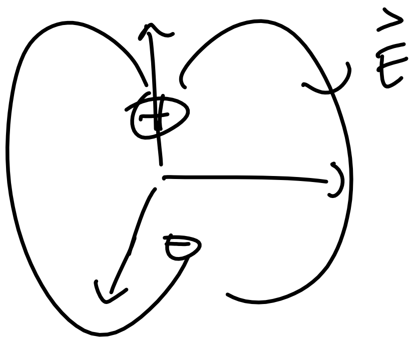
$$V = \frac{q s \cos\vartheta}{4\pi \epsilon_0 r^2}$$



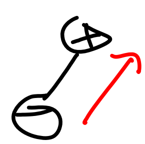
$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\cos\vartheta = \frac{z}{|\vec{r}|}$$

$$V = \frac{qs}{4\pi\epsilon_0} \frac{z}{(x^2+y^2+z^2)^{3/2}} \qquad \vec{E} = -\vec{\nabla} V$$



3.) $\vec{F} = q(\vec{E}_+ - \vec{E}_-) = q\Delta\vec{E}$



$$\vec{p} = q\vec{s}$$

$$\vec{E}(x,y,z) = E_x(x,y,z)\hat{x} + E_y(x,y,z)\hat{y} + E(x,y,z)\hat{z}$$

$$dE_x = \frac{\partial E_x}{\partial x}dx + \frac{\partial E_x}{\partial y}dy + \frac{\partial E_x}{\partial z}dz$$

$$dE_x = \left(\frac{\partial E_x}{\partial x}\hat{x} + \frac{\partial E_x}{\partial y}\hat{y} + \frac{\partial E_x}{\partial z}\hat{z}\right)\cdot\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right)$$

$$\rightarrow \vec{\nabla} E_x \cdot \vec{s}$$

since $d\vec{E} = dE_x\hat{x} + dE_y\hat{y} + dE_z\hat{z} = \vec{\nabla}\vec{E}\cdot\vec{s}$

or $$d\vec{E} = (\vec{s}\cdot\vec{\nabla})\vec{E}$$

$$q\,d\vec{E} = (q\vec{s}\cdot\vec{\nabla})\vec{E} = (\vec{p}\cdot\vec{\nabla})\vec{E}$$

4.)

$$\vec{\mathscr{r}} = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$$

$$|\vec{\mathscr{r}}| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\vec{\nabla}\frac{1}{|\vec{\mathscr{r}}|} = \frac{\partial}{\partial x}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-1/2}\hat{x} + \cdots$$

$$= -\frac{1}{2}[\ ]^{-3/2}\,2(x-x')\hat{x} - \frac{1}{2}[\ ]^{-3/2}\,2(y-y')\hat{y} - \frac{1}{2}[\ ]^{-3/2}\,2(z-z')\hat{z}$$

$$= -[\ ]^{-3/2}\,(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z} = -\frac{1}{|\vec{\mathscr{r}}|^3}\vec{\mathscr{r}} = \frac{\hat{\mathscr{r}}}{|\vec{\mathscr{r}}|^2}$$

5.)

$$V = \int dV = \frac{1}{4\pi\epsilon_0}\int \vec{P}\cdot\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)dx'dy'dz'$$

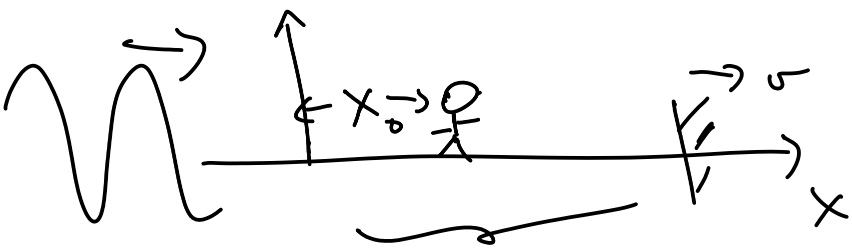
$$\vec{\nabla}\cdot(f\vec{A}) = f\vec{\nabla}\cdot\vec{A} + A\cdot\vec{\nabla}f$$

$$\frac{1}{4\pi\epsilon_0}\int \vec{P}\cdot\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)dx'dy'dz' = \frac{1}{4\pi\epsilon_0}\int\left[\vec{\nabla}\cdot\frac{\vec{P}}{|\vec{r}-\vec{r}'|} - \frac{1}{|\vec{r}-\vec{r}'|}\vec{\nabla}\cdot\vec{P}\right]d\tau'$$

divergence theorem

$$V = \frac{1}{4\pi\epsilon_0}\oint \frac{\vec{P}\cdot d\vec{a}'}{|\vec{r}-\vec{r}\,'|} + \frac{1}{4\pi\epsilon_0}\int \frac{-\vec{\nabla}'\cdot\vec{P}}{|\vec{r}-\vec{r}\,'|}\,dx'dy'dz'$$

6.) Homework problem 6.) Derive the expression for the wave reflected from a mirror moving at constant speed using the retarded time method.



$$\varphi(x=0,t) = -\omega t$$

$$\varphi(x_0,t) = ?$$

$$x_{mirror} = L + vt$$

$$\text{Let} \quad x_{mirror}(t) = L + vt$$

$$t_{retarded} = t - t_{out} - t_{return}$$

now

t_{out} is the time it took the wavecrest, which reaches us now, to go from the origin to the mirror.

t_{return} is the time it took the wavecrest, which reaches us now, to go from the mirror to our location.

$$L(t - t_{return}) - x_0 = c\,t_{return}$$

The LHS of the above expression is the distance the reflected wavecrest has to travel. The RHS (not shown) is another expression for this distance the crest travels (at speed c) but for what time?

We now need an equation for the time out. The crest strikes the mirror when the mirror is located at

$$L(t - t_{return}) = c\,t_{out}$$

These are two eqns in the two unknowns t_{return} & t_{out}

when you substitute $L(t - t_{return}) = L_0 + v(t - t_{return})$

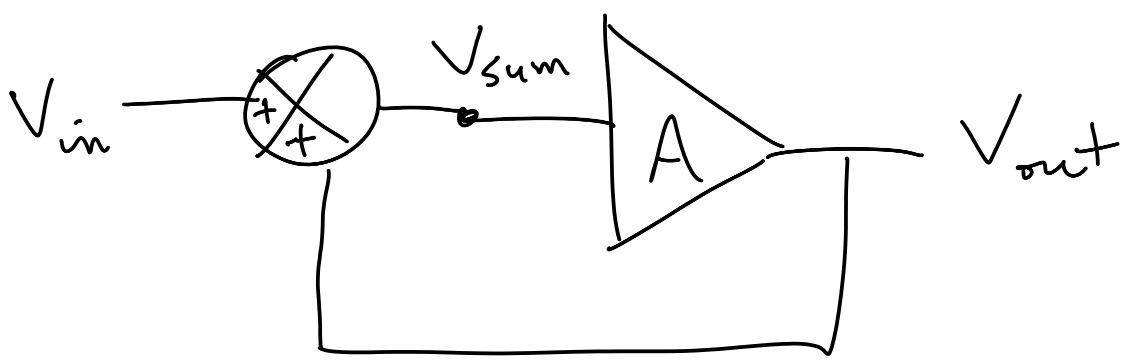
and solve you get $t_{return} = \dfrac{L_0 + vt - x_0}{c+v}$; $t_{out} = \dfrac{L_0 + vt - \frac{v}{c}x_0}{c+v}$

The phase is $\varphi = -\omega(t - t_{return} - t_{out})$

The electric field is $E = E_0 e^{-i\varphi}$

The reflected frequency is $\omega_{reflected} = \dfrac{d\varphi}{dt} = \omega\left(\dfrac{c-v}{c+v}\right)$

7.) Circuit problem: A positive feeback amplifier's block diagram is



$$V_{sum} = V_{in} + V_{out}$$

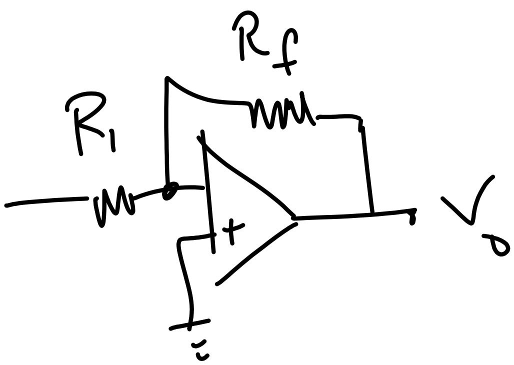
pos feedbach

$$V_{out} = A V_{sum} = A(V_{in} + V_{out})$$

single pass gain

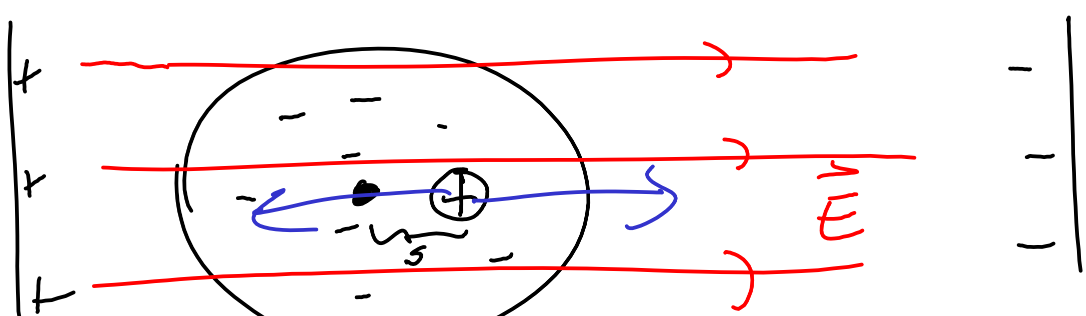
$$\Rightarrow \frac{V_{out}}{V_{in}} = \frac{A}{1-A} \simeq A(1 + A + A^2 + \dots)$$

A is the single pass gain NOT the total gain as calculated in



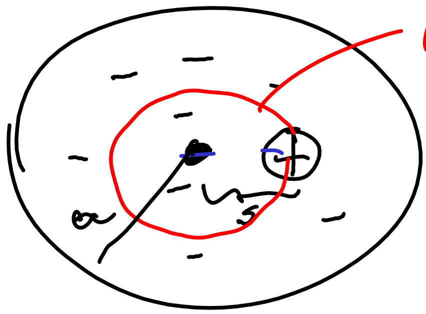
$$A_{tot} = -\frac{R_f}{R_1} \qquad \frac{V_o}{V_{in}} = A_{tot}$$

8.)



$$p = qs$$

$$\Sigma \vec{F} = q E_{electron\ cloud} - q E_{applied} = 0$$



Gaussian surface to determine E at the proton

$$\oint \vec{E}_{electron}\cdot d\vec{a} = \frac{Q_{enclosed}}{\epsilon_0}$$

$$E_{electron}\, 4\pi s^2 = \frac{\frac{q}{\frac{4\pi}{3}a^3}\,\frac{4\pi}{3}s^3}{\epsilon_0}$$

$$E_{electron} = \frac{q}{4\pi\epsilon_0}\frac{s}{a^3} = E_{applied}$$

$$p = \alpha E_{applied}$$

$$\parallel$$

$$qs = \alpha \frac{q s}{4\pi\epsilon_0 a^3} \Rightarrow \alpha = 4\pi\epsilon_0 a^3$$

9.) $U = qV_2 - qV_1 = q\Delta V$

Homework problem 9: Find an expression for the potential energy U in two ways:

(1) use $\Delta V = \int_{r_1}^{r_2} \underbrace{\vec{\nabla} V \cdot d\vec{r}}_{\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz}$

$U = q\vec{\nabla}V\cdot\vec{s}$ where $\vec{s} \longleftrightarrow d\vec{r}$

$= q\vec{s}\cdot\vec{\nabla}V$ since $E = -\vec{\nabla}V$ $\quad U = -\vec{p}\cdot\vec{E}$

(2) $\vec{F} = -\vec{\nabla}(\text{Potential Energy})$

and a previous homework result that

$$\vec{F} = (\vec{p} \cdot \vec{\nabla})\vec{E}$$

Use vector identity $(\vec{A} \cdot \vec{\nabla})\vec{B} = \vec{\nabla}_B(\vec{A} \cdot \vec{B}) - \vec{A} \times (\vec{\nabla} \times \vec{B})$

$\vec{p}$ is constant

∇_B operates only on B

$$\vec{F} = \vec{\nabla}(\vec{p} \cdot \vec{E}) - \vec{p} \times (\underbrace{\vec{\nabla} \times \vec{E}}_{=0})$$

$$\vec{F} = -\vec{\nabla} U$$

$$\Rightarrow \quad U = -\vec{p} \cdot \vec{E}$$

10.) $n(\theta) \equiv$ number of molecules per unit solid angle per volume

$$n(\theta) = n_0 e^{-U/kT} = n_0 e^{p_0 E \cos\theta / kT}$$

$$\simeq n_0 \left(1 + \frac{p_0 E \cos\theta}{kT}\right) \quad \text{since } e^{\delta} \simeq 1 + \delta + \cdots$$

$$N = \int n(\theta)\, d\Omega = \int_0^{2\pi}\int_0^{\pi} n_0 \left(1 + \frac{p_0 E \cos\theta}{kT}\right) \underbrace{\sin\theta\, d\theta\, d\varphi}_{d\Omega}$$

$$N = n_0 4\pi$$

so

$$n = \left(1 + \frac{p_0 E \cos\theta}{kT}\right) \frac{N}{4\pi}$$

$$\vec{P} = \int n(\theta)\, p_0 \cos\theta\, d\Omega = \int_0^{2\pi}\int_0^{\pi} \frac{N}{4\pi}\left(1 + \frac{p_0 E \cos\theta}{kT}\right) p_0 \cos\theta \sin\theta\, d\theta\, d\varphi$$

$$\simeq \frac{N p_0^2}{3kT} E$$

Note that pE has units of kT so we are left with only N p.