

10-25-07

Note Title

10/25/2007


Fourier Transforms

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

ex. $f(t) = \delta(t)$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \underbrace{e^{-i\omega t}}_{\left. \vphantom{e^{-i\omega t}} \right|_{t=0} = 1}$$


$$f(t) = \delta(t - t')$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t - t') e^{-i\omega t} dt$$
$$= \frac{1}{\sqrt{2\pi}} \underbrace{e^{-i\omega t}}_{\left. \vphantom{e^{-i\omega t}} \right|_{t=t'}}$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} e^{-i\omega t'}$$

$$\operatorname{Re}[F(\omega)] = \frac{1}{\sqrt{2\pi}} \cos \omega t'$$

$$\operatorname{Im}[F(\omega)] = \frac{1}{\sqrt{2\pi}} \sin \omega t'$$

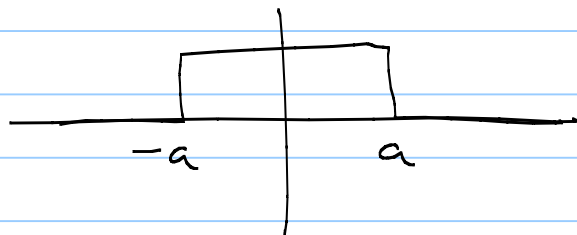
$$|F(\omega)|^2 = \frac{1}{2\pi}$$

important point

Fourier Transform of $\delta(t-t')$
= 1 if and only if $t'=0$

But power spectrum of $\delta(t-t')$
is always 1

EX. $f(x) =$



1 if $-a \leq x \leq a$
0 otherwise

$$x \leftrightarrow k \quad t \leftrightarrow \omega$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a e^{-ikx} dx$$

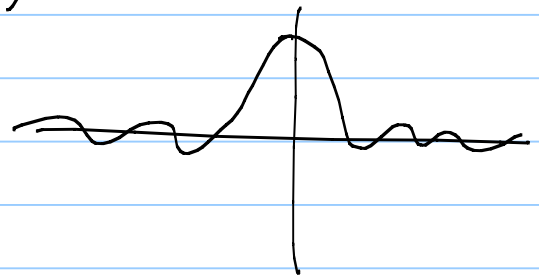
$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-ik} e^{-ikx} \Big|_{-a}^a \right]$$

$$= \frac{i}{\sqrt{2\pi} k} \left[(\cos ka - i \sin ka) - (\cos ka + i \sin ka) \right]$$

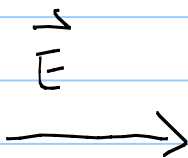
$$\begin{aligned} \cos(-x) &= \cos x \\ \sin(-x) &= -\sin x \end{aligned}$$

$$= \frac{i}{\sqrt{2\pi} k} (-2i \sin ka)$$

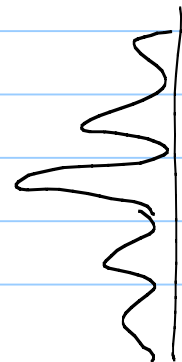
$$\frac{2}{\sqrt{2\pi}} \frac{\sin ka}{k}$$



Claim: $|F(k)|^2$ is the diffraction pattern of a rectangular slit.

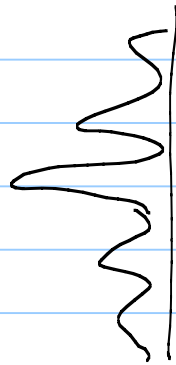


very long thin slit



Plot

$$\frac{\sin^2 ka}{k^2}$$

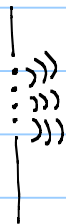


} becomes k
axis in the
experiment

Physics

Screen obstructs 100% of the light except for the slit. Imagine each point in the slit is a source of secondary wavelets
- Huyghen's principle

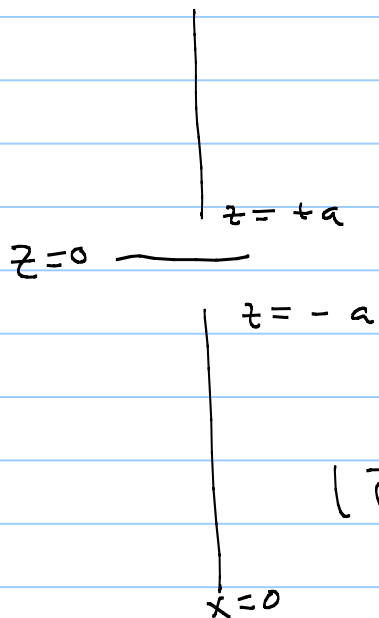
mathematics



each little point source
is a spherical wave
$$\frac{e^{ik(r-r')}}{|r-r'|}$$

So field on the screen is the superposition of all of these point sources

E field at a point r' (on screen) is


$$\int_{-a}^a \frac{e^{ik(r-r')}}{|r-r'|} dz$$

$$|\vec{r}-\vec{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

if the slit is small and the distance to the screen is large, then the integral can be approximated. [see references on the wiki] Far field approx is called **Fraunhofer diffraction.**

Fourier Transform of a Gaussian

Lemma $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

Proof Let $I = \int_{-\infty}^{\infty} e^{-x^2} dx$

$$I^2 = \left[\int_{-\infty}^{\infty} e^{-x^2} dx \right] \left[\int_{-\infty}^{\infty} e^{-y^2} dy \right]$$

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dummy var. of integ.

recast as double integral

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy$$

$$= \int_0^{\infty} \int_0^{2\pi} e^{-r^2} r dr d\theta$$

$$= 2\pi \int_0^{\infty} e^{-r^2} r dr$$

Let $\rho = r^2$ $d\rho = 2r dr$

$$I^2 = 2\pi \int_0^{\infty} e^{-\rho} \frac{d\rho}{2}$$

$$= \pi \left[-e^{-\rho} \Big|_0^{\infty} \right] = \pi$$

$$\Rightarrow I = \sqrt{\pi}$$