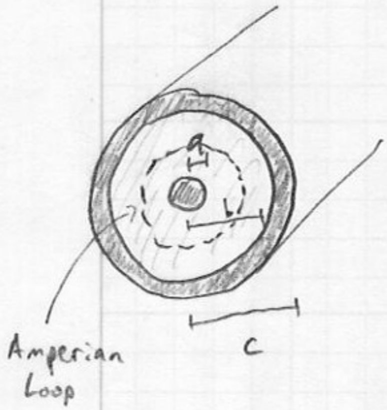
Day 27: Examples, including the magnetic scalar potential

One problem we've solved in the past is the electric field in a coax cable with a dielectric spacer. Let's solve the analogous problem in magnetics.



Coax cable. Interior wire has radius a. Return sheath has inner and outer radii b + c.

The gap is filled with an insulator with magnetic susceptibility χ_m.

If there's a current I, we have free current densities

$$J_{f_{in}} = I/\pi a^2 \quad \text{for} \quad r<a \qquad \text{and}$$

$$J_{f_{out}} = I/\pi(c^2-b^2) \quad \text{for} \quad b<r<c$$

With the J vectors in opposite directions.

We can't use $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ because $\vec{J}$ includes bound currents, so we instead work from $\vec{\nabla}\times\vec{H} = J_f$ or $\oint \vec{H}\cdot d\vec{l} = I_{free,enc}$

To find H we draw an Amperian loop in one of the four regions:

1. $r<a$: $\oint \vec{H}\cdot d\vec{l} = I_f \Rightarrow H\cdot 2\pi r = J_{f,in}\,\pi r^2 = \dfrac{I r^2}{a^2}$

$$\Rightarrow \boxed{\vec{H} = \frac{Ir}{2\pi a^2}\hat{\phi} \quad r<a, \quad \vec{J} \text{ in the } \hat{k} \text{ direction}}$$

2. $a<r<b$: $I_{enclosed}$ is the entire I, so

$$\boxed{\vec{H} = \frac{I}{2\pi r}\hat{\phi} \quad a<r<b}$$

3. $b<r<c$: We enclose some current going in both directions, so

$$I_{enc} = I - J_{f,out}\cdot(\pi r^2 - \pi b^2) = I - I\frac{(r^2-b^2)}{c^2-b^2} = I\left[\frac{c^2-b^2}{c^2-b^2} - \frac{(r^2-b^2)}{(c^2-b^2)}\right]$$

$$\Rightarrow \boxed{\vec{H} = \frac{I(c^2-r^2)}{2\pi r(c^2-b^2)}\hat{\phi} \quad b<r<c}$$

4. $r>c$: We now enclose all the current in both directions, so $I_{enc}=0$.

$$\Rightarrow \boxed{\vec{H} = 0 \quad r>c}$$

Note that throughout this problem we've been taking advantage of the cylindrical symmetry of the situation and the knowledge that $\vec{H}$ had to point in the $\hat{\phi}$ direction.

Ok, so let's find everything else. $\vec{B} = \mu \vec{H}$, and $\mu = \mu_0$ in regions 1, 3, + 4 and $(1+\chi_m)\mu_0$ in region 2, so

$$\vec{B} = \quad \frac{\mu_0 I r}{2\pi a^2}\hat{\phi} \quad (1) \qquad \frac{\mu_0(1+\chi_m) I}{2\pi r}\hat{\phi} \quad (2)$$

$$\frac{\mu_0 I (c^2 - r^2)}{2\pi r (c^2 - b^2)}\hat{\phi} \quad (3) \qquad 0 \qquad (4)$$

And the magnetization $\vec{M} = \chi_m \vec{H}$ and so is 0 in 1, 3, 4 and

$$\vec{M} = \frac{\chi_m I}{2\pi r}\hat{\phi} \quad (2)$$

And that's all there is to know.

The magnetic scalar potential

You may recall that $\vec{\nabla}\times\vec{E} = 0$ implies $\oint \vec{E}\cdot d\vec{l} = 0$, which implies that $\int \vec{E}\cdot d\vec{l}$ is path independent and we can use it to build a scalar potential function V such that $\vec{E} = -\vec{\nabla}V$.

This doesn't work so well for $\vec{B}$ since $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$ and there's usually nonzero $\vec{J}$ if we're bothering to talk about $\vec{B}$. But, there's a loophole.

$\vec{\nabla}\times\vec{H} = \vec{J}_f$ and sometimes $\vec{J}_f$ is zero when $\vec{J}_b$ isn't, so

$\vec{\nabla}\times\vec{B} \neq 0$ but $\vec{\nabla}\times\vec{H} = 0$ and we have a legit situation where we can use a scalar potential for $\vec{H}$:

$$\vec{H} = -\vec{\nabla}\phi_m$$ This works anytime $\vec{J}_f$ is zero.

But why do we bother? What does this gain us? Well,

$\vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot(\mu\vec{B})$ so if μ is spatially uniform, $\vec{\nabla}\cdot\vec{H} = 0$, so $\vec{\nabla}\cdot(-\vec{\nabla}\phi_m) = 0$ and

$$\boxed{\nabla^2\phi_m = 0}$$

OH SNAP!
We know all kinds of dirty tricks for solving this equation.

As Pollack says, we can "bring to bear" the "considerable mathematical armamentarium" we have for such problems. In other words, we just made magnetostatics look like electrostatics (if $\vec{J}_f = 0$).

This really works, too. Check out an example:

A sphere of radius a made of a material with susceptibility χ_m placed in a uniform magnetic field $\vec{B} = B_0\hat{k}$.

There is no J_f, so we can solve $\vec{\nabla}^2 \Phi_m = 0$ and go nuts. Which the book does. But you know what's better? <u>We already solved this problem.</u>

Back in chapter 6 we did a polarizable sphere in a uniform $\vec{E}$-field. This is structurally identical. And what does a physicist do when they see a problem they've already solved? They write down the answer.

From page 210:
$$V_{inside} = -C_1 r\cos\theta \qquad \text{With } C_1 = \frac{3}{K+2}E_0$$
$$V_{outside} = -E_0 r\cos\theta + \frac{C_2 a^3}{r^2}\cos\theta \qquad C_2 = \frac{K-1}{K+2}E_0$$

Now we just fix it up a little. K is the dielectric "constant", $\varepsilon/\varepsilon_0$, so replace it with $K_m = \mu/\mu_0$. And in electrostatics $\vec{E} = -\nabla V$, but in magnetostatics $\vec{H} = -\nabla\Phi_m$, so replace E_0 with H_0, <u>not</u> B_0. And $H_0 = B_0/\mu_0$, so we get:

$$\Phi_{m\,in} = \frac{-3B_0}{(K_m+2)\mu_0} r\cos\theta \quad \text{and} \quad \Phi_{m\,out} = -\frac{B_0}{\mu_0} r\cos\theta + \left(\frac{K_m-1}{K_m+2}\right)\frac{B_0 a^3}{r^2}\cos\theta$$

Just like that. Now, there are some major <u>physical</u> difference between magnetostatics and electrostatics. In electrostatics $K \geq 1$ always, but in magnetostatics K_m can be less than 1.

Also, in electrostatics, polarizing an object with $K > 1$ leads to a charge separation that opposes the applied field. But in magnetostatics, magnetizing an object with $K_m > 1$ enhances the applied field.

Let's see how this plays out in the above example for the interior E+B.

For $\vec{E}$: $V_{inside} = -\frac{3E_0}{K+2} r\cos\theta \Rightarrow \vec{E}_{inside} = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta}$
$$= +\frac{3E_0}{K+2}\cos\theta\,\hat{r} - \frac{3E_0}{K+2}\sin\theta\,\hat{\theta}$$

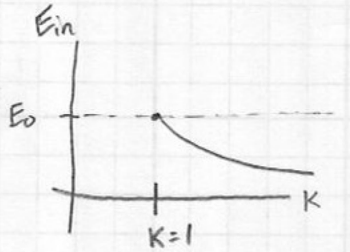
And $\hat{k} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$, so
$$\boxed{\vec{E}_{inside} = \frac{3E_0}{K+2}\hat{k}}$$

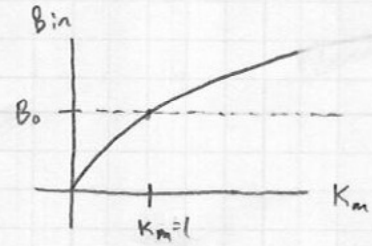
Now, we can see immediately that $\vec{H}_{in} = \frac{3H_0}{K_m+2}\hat{k}$ And $\vec{B} = \mu\vec{H}$

So $\frac{\vec{B}_{in}}{\mu} = \frac{3B_0}{(K_m+2)\mu_0}\hat{k}$ And since $K_m = \mu/\mu_0$

$$\boxed{\vec{B}_{in} = \frac{3K_m}{(K_m+2)} B_0 \hat{k}}$$

Sketching these out, we see



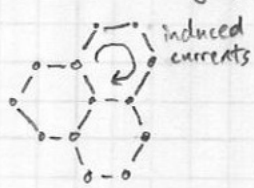


$K_m > 1$ indicates a paramagnetic material.
$K_m < 1$ indicates a diamagnetic material.

Strongly diamagnetic materials significantly exclude applied B-fields.

Superconductors have $K_m = 0$ and repel $\vec{B}$ completely, much as conductors have $K \to \infty$ so that $\vec{E}_{inside} = 0$.

Demo: Pyrolitic graphite: A form of graphite that consists of sheets of graphene, planar arrangements of carbon atoms:



It is more diamagnetic at room temperature than any other known material.

A semiclassical explanation of this is that the carbon rings allow for strong induced currents when a $\vec{B}$-field is brought near. By Lenz's law, those currents will be in such a way as to create B-fields that oppose the change in B that induced the currents, leading to the possibility of levitation

Magnetic Shielding

I probably won't have to work very hard to convince you that sometimes you'll want to keep external magnetic fields away from certain objects or devices. Let's look at the principal of magnetic shielding.

Consider a <u>spherical</u> shell of material with permeability μ and inner and outer radii R_i and R_o.

We apply a magnetic field of the form $\vec{B} = B_0 \hat{k}$

There are no free currents, so we can define $\vec{H} = -\nabla \Phi_m$

The external $\vec{H}$ is B/μ_0, so $\Phi_{m,external} = -B_0/\mu_0 \, z = -B_0/\mu_0 \, r\cos\theta$

We've solved problems like this before. Basically we looked at the solution to $\nabla^2 V = 0$ in spherical and pulled out the terms proportional to $\cos\theta$. Those went like $r\cos\theta$ and $\cos\theta/r^2$.

For $r < R_i$, we exclude $\cos\theta/r^2 \Rightarrow \boxed{\Phi_{m,in}(r,\theta) = \alpha r\cos\theta}$

For $R_i < r < R_o$, $\boxed{\Phi_{m,mid}(r,\theta) = \gamma r\cos\theta + \delta \frac{\cos\theta}{r^2}}$

For $r > R_o$, we know the $r\cos\theta$ term must match the applied potential at large r, so

For $r > R_o$, $\boxed{\Phi_{m,out}(r,\theta) = -B_0/\mu_0 \, r\cos\theta + \beta \frac{\cos\theta}{r^2}}$

Four constants to find: $\alpha, \beta, \gamma, \delta$. We need to invoke four BCs. There are many possible choices; as always there's a bit of an art to choosing easy ones. I, for one, like to use the continuity of the potential whenever possible. The book seems to agree.

$$\Phi_{in}(R_i) = \Phi_{mid}(R_i) \Rightarrow \alpha R_i = \gamma R_i + \delta/R_i^2$$

$$\Phi_{mid}(R_o) = \Phi_{out}(R_o) \Rightarrow \gamma R_o + \delta/R_o^2 = -B_0/\mu_0 + B/R_o^2$$

And we can get two more equations from the continuity of $B_\perp$:

$$\mu_0 \frac{\partial \Phi_{in}(R_i)}{\partial r} = \mu \frac{\partial \Phi_{mid}(R_i)}{\partial r} \quad \text{and} \quad \mu \frac{\partial \Phi_{mid}(R_o)}{\partial r} = \mu_0 \frac{\partial \Phi_{out}(R_o)}{\partial r}$$

We end up with four linear equations in four unknowns.
Matrix mojo follows.

The complete list of coefficients is in the book. We'll just look at α since it's pertinent to B_{inside}:

$$\alpha = \frac{-9K_m}{(2K_m+1)(K_m+2) - 2(R_i/R_o)^3(K_m-1)^2} \, B_0/\mu_0$$

For large K_m, the denominator is $\propto K^2$, so $\alpha \propto 1/K_m$

and $B_{inside} \propto 1/K_m$

The more permeable the material, the weaker the field in the shielded region. There exist materials engineered to have very high K_m, such as the alloy Mu-metal (Fe, Ni, Cu, Cr) with K_m up to 10^5

(slides)