Lecture 12

Note Title 2/8/2006

$\nabla^2 V = 0$ with bndy conditions

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$xy + x^2$ doesn't stay there

Assumption $V(x,y,z) = X(x)\,Y(y)\,Z(z)$

plug $V(x,y,z)$ into Laplaces eqn

$$\underbrace{\frac{1}{X(x)}\frac{d^2 X}{dx^2}} + \underbrace{\frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2}} + \underbrace{\frac{1}{Z}\frac{d^2 Z}{dz^2}} = 0$$

$$g(x) + h(y) + k(z) = 0$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$c_1 + c_2 + c_3 = 0$$

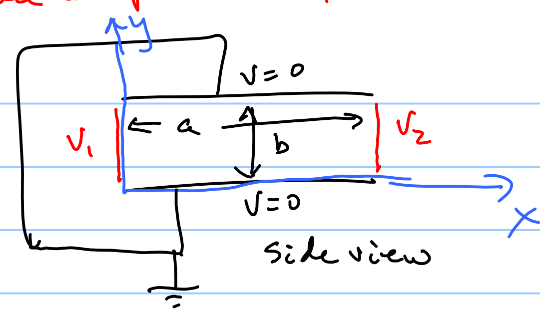
Started with P.D.E. & now have 3 O.D.E.'s

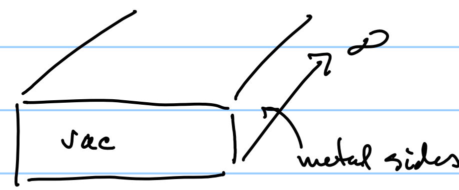
$$\frac{1}{V}$$

Ex: 2 gnded ||-plate electrodes

other plates are at potential V_1 & V_2 (constant)

BC $\begin{cases} x = 0 & V = V_1 \\ x = a & V = V_2 \\ y = 0, b; & V = 0 \end{cases}$





Field & potential in z direction: do not depend on z

$Z_1(z) = \text{constant} \Rightarrow c_3 = 0$

$$\frac{d^2 Y}{dy^2} = c_2 Y$$

eigenvalue eqn.

$c_2 < 0 \quad Y = A\sin(ky) + B\cos(ky) \quad c_2 = -k^2$

$c_2 > 0 \quad Y = A' e^{\sqrt{c_2}\, y} + B' e^{-\sqrt{c_2}\, y}$

$c_2 = 0 \quad Y = A'' + B'' y$

Bndry at $y = 0$ & $y = b$ is $V = 0$ so $\Rightarrow$ $c_2 < 0$ with $k^2 > 0$

$$\underline{Y} = A\sin ky + B\cos ky \quad \begin{cases} \underline{Y}(y=0) = 0 = 0 + B\cos 0 = B \\ \underline{Y}(y=b) = A\sin kb = 0 \end{cases}$$

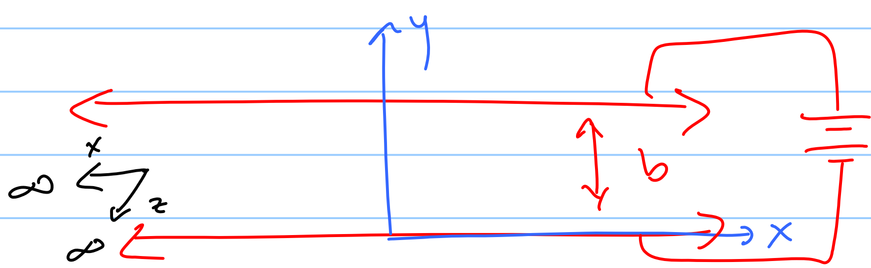
$$kb = n\pi \quad \underline{n = 1, 2, 3, \dots}$$

$$\boxed{k = \frac{n\pi}{b}}$$

$$\boxed{\underline{Y} = A\sin\left(\frac{n\pi}{b}y\right)} \qquad \boxed{Z_1 = \text{constant}}$$

$$\underset{-k^2}{C_1} + \underset{0}{C_2} + C_3 = 0 \Rightarrow C_1 = k^2 > 0$$

$$\frac{d^2\underline{X}}{dx^2} = k^2\underline{X} \qquad \boxed{\underline{X}(x) = Ge^{kx} + He^{-kx}}$$



$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = C_3 = 0 \quad (\text{const})$$

$$\frac{1}{\underline{X}}\frac{d^2\underline{X}}{dx^2} = C_1 = 0 \quad (\text{const})$$

$$\underset{0}{C_1} + \underset{?}{C_2} + \underset{0}{C_3} = 0$$

$C_2 = 0 \Rightarrow$ Linear $\underline{\text{soln}}$

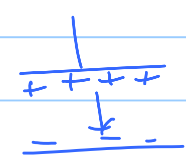
$$V = XYZ = (A + 0x)(A' + 0z)(A'' + By)$$

$$V = \underbrace{AA'A''}_{a} + \underbrace{AA'B}_{b}\, y$$

$$V = a + by$$

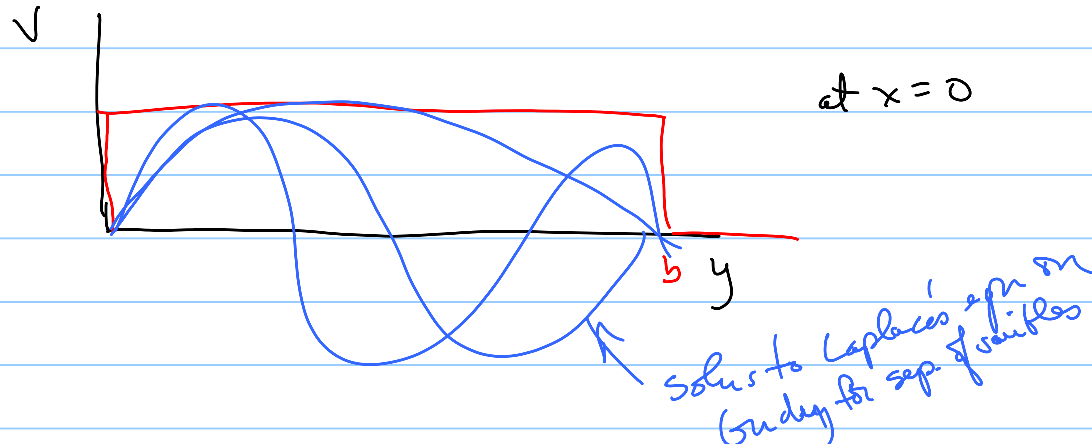
$$V = Ay + B$$

$$\vec{E} = -\vec{\nabla} V = -A\hat{y} \quad \text{const}$$



Back to our 1st problem

$$V = XYZ = \left(G e^{kx} + H e^{-kx}\right) \sin ky \; ; \; k = \frac{n\pi}{b}$$



Laplace's eqn obeys superposition ⇒ sum of solns is a soln

$$V = \sum_{i=1}^{N} A_i V_i$$ also a soln also a Fourier series

$$V = \sum_{n=1}^{\infty} \left(A_n e^{-\frac{n\pi x}{b}} + B_n e^{n\pi x/b}\right) \sin\left(\frac{n\pi y}{b}\right)$$ n is integer

A_n & B_n are determined by bndry cond

$$V(x=0) = V_1 = \sum_1 (A_n + B_n) \sin\left(\frac{n\pi y}{b}\right)$$

multiply by $\sin \frac{m\pi y}{b}$ & integrate from $0 \to b$

$$V_1 \int_0^b \sin\left(\frac{m\pi y}{b}\right) dy = \sum_{n,1} (A_n + B_n) \underbrace{\int_0^b \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{m\pi y}{b}\right) dy}$$

$$V_1 \left[-\frac{b}{m\pi} \cos\left(\frac{m\pi y}{b}\right) \right]_0^b = \sum_n (A_n + B_n) \frac{b}{2} \delta_{nm}$$

$$-\frac{bV_1}{m\pi} (\cos(m\pi) - 1) = (A_m + B_m) \frac{b}{2}$$

$$\boxed{\begin{array}{lll} m \text{ odd} & \frac{2bV_1}{m\pi} & = (A_m + B_m)\frac{b}{2} \\ m \text{ even} & 0 & = (A_m + B_m)\frac{b}{2} \end{array}}$$

1 eqn in 2 unknowns

Other bndy

$$V(a) = V_2 = \sum_{n,1} \left(A_n e^{-\frac{n\pi a}{b}} + B_n e^{\frac{n\pi a}{b}} \right) \sin\left(\frac{n\pi y}{b}\right) \Big\} \times \sin\frac{m\pi y}{b} dy$$

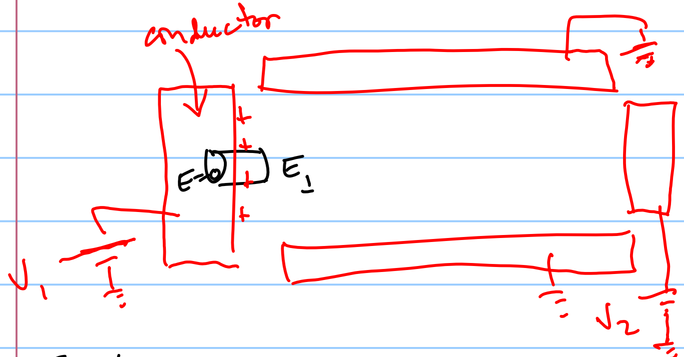
$$\Rightarrow \boxed{A_n e^{-n\pi a/b} + B_n e^{n\pi a/b} = \begin{cases} \frac{4V_2}{n\pi} & n = \text{odd} \\ 0 & n = \text{even} \end{cases}}$$

2 eqns in A_n & B_n

$$A_n = \frac{4}{n\pi}\left(\frac{V_1 - V_2 e^{-n\pi a/b}}{1 - e^{-2n\pi a/b}}\right) \qquad A_n = -B_n$$

final soln

$$V = \sum_{n=\text{odd}}^{\infty}\left(A_n e^{-\frac{n\pi x}{b}} + B_n e^{\frac{n\pi x}{b}}\right)\sin\left(\frac{n\pi y}{b}\right)$$

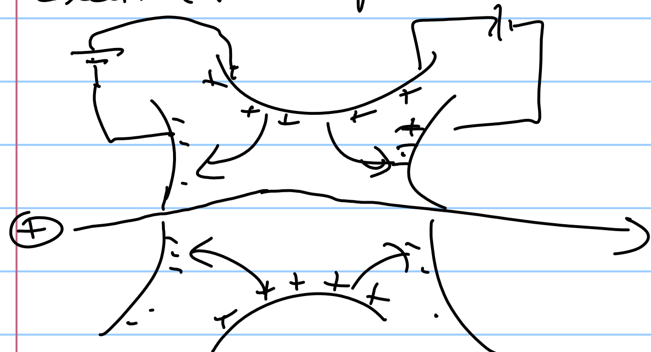


$$\overset{0}{E_{in}} A + E_{out} \cancel{A} = \frac{Q_{enc}}{\epsilon_0 \cancel{A}}$$

$$\downarrow \qquad \downarrow$$

$$\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\epsilon_0}$$

Electron microscope



$$\vec{F} = q\vec{E}$$

use relaxation method to solve (Excel spreadsheet) for V. Then get σ from $\frac{\partial V}{\partial n}$