# Dispersion

Connections: susceptibility, dielectric constant, refractive index

Microscopic to macroscopic material response

Review of classical oscillator model for dispersion

Complex refractive index, damped propagation

# Maxwell's Equations in a Medium

- The induced polarization, **P**, contains the effect of the medium:

$$\vec{\nabla}^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}$$

- Sinusoidal waves of all frequencies are solutions to the wave equation
- For linear response, **P** will oscillate at the same frequency as the input.

$$\mathbf{P}(t) = \varepsilon_0 \chi \mathbf{E}(t) \qquad \rightarrow \vec{\nabla}^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \varepsilon_0 \chi \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

- Then once we know the susceptibility χ, we can calculate the dielectric constant and the refractive index:

$$\rightarrow \vec{\nabla}^2\mathbf{E} - \frac{1}{c^2}(1+\chi)\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\varepsilon_0(1+\chi) = \varepsilon_0 \varepsilon_r = \varepsilon_0 n^2$$

$$\vec{\nabla}^2\mathbf{E} - \frac{n^2}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

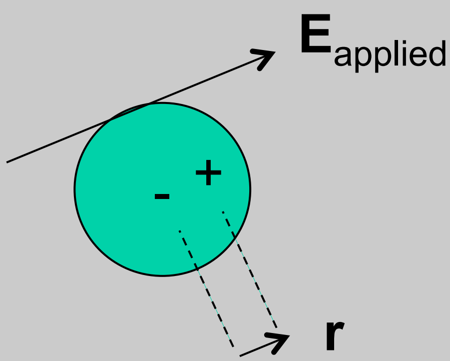
# Connecting the macroscopic to the microscopic response

So determining the gain or loss coefficient depends on calculating the macroscopic induced polarization **P** or equivalently the susceptibility χ.

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \chi \mathbf{E} = N_a \mathbf{p}$$

Note that the *macrosopic* polarization is really a density of individual dipole moments on the *microscopic* scale.

Recall: $\mathbf{p} = q\,\mathbf{r}$




So if the electric field is linearly polarized in the x-direction, then

$$\mathbf{P}(t) = N_a \mathbf{p}(t) = -N_a e\,x(t)$$

Here we treat x(t) as the position of the *electron*.

# Spring model for dipole response

- Model: driven SHO with damping

$$m_e\ddot{x}(t) = -eE(t) - m_e\omega_0^2 x(t) - 2m_e\gamma\dot{x}(t)$$

$$m_e\ddot{x}(t) + m_e\gamma\dot{x}(t) + m_e\omega_0^2 x(t) = -eE_0 e^{-i\omega t}$$

Radiation damping term $\gamma$

Restoring force, resonant at $\omega_0$

External driving field, specific $\omega$

let $x(t) = x_0 e^{-i\omega t}$ — *x* must oscillate at driving frequency, not at resonance frequency

$$-m_e\omega^2 x_0 - i\omega m_e\gamma x_0 + m_e\omega_0^2 x_0 = -eE_0$$

$e^{-i\omega t}$ drops out

$$x_0(\omega) = -\frac{e}{m_e}E_0\frac{1}{\omega_0^2 - i\omega\gamma - \omega^2} \equiv -\frac{eE_0}{m_e}\frac{1}{D(\omega)}$$

*x* is fcn of *t*, but the amplitude $x_0$ depends on $\omega$

# Spring model for dispersion

- Now we can go from the microscopic response *x*(*t*) to the macroscopic *χ* and *n*

$$P(t) = -N_a e\,x(t) = \varepsilon_0 \chi^{(1)} E(t)$$

$$\rightarrow \chi^{(1)} = -\frac{N_a e\,x(t)}{\varepsilon_0 E(t)} = -\frac{N_a e\,x_0(\omega) e^{-i\omega t}}{\varepsilon_0 E_0 e^{-i\omega t}}$$

$$x_0(\omega) = -\frac{e}{m_e} E_0 \frac{1}{\omega_0^2 - i\omega\gamma - \omega^2} \equiv -\frac{e E_0}{m_e} \frac{1}{D(\omega)}$$

$$\chi^{(1)}(\omega) = -\frac{N_a e}{\varepsilon_0} \left( -\frac{e}{m_e} \frac{E_0}{D(\omega)} \right) \frac{1}{E_0} = \frac{N_a e^2}{\varepsilon_0 m_e} \frac{1}{D(\omega)}$$

Note that this gives us the *frequency response* of the system.

# Spring model: refractive index

- Linear susceptibility yields the refractive index:

$$n^2(\omega) = 1 + \chi^{(1)}(\omega) = 1 + \frac{N_a e^2}{\varepsilon_0 m_e} \frac{1}{D(\omega)}$$

$$n^2(\omega) = 1 + \frac{N_a e^2}{\varepsilon_0 m_e} \frac{1}{\omega_0^2 - i\omega\gamma - \omega^2}$$

Refractive index is a *complex* quantity.

- Solve for the Re and Im parts:

$$n^2(\omega) = 1 + \frac{N_a e^2}{\varepsilon_0 m_e} \frac{\omega_0^2 - \omega^2 + i\omega\gamma}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2} = 1 + \frac{N_a e^2}{\varepsilon_0 m_e} \frac{(\omega_0 + \omega)(\omega_0 - \omega) + i\omega\gamma}{(\omega_0 + \omega)^2(\omega_0 - \omega)^2 + \omega^2\gamma^2}$$

- Near a resonance, $\omega_0 + \omega \approx 2\omega_0$ $\quad \omega\gamma \approx \omega_0\gamma$

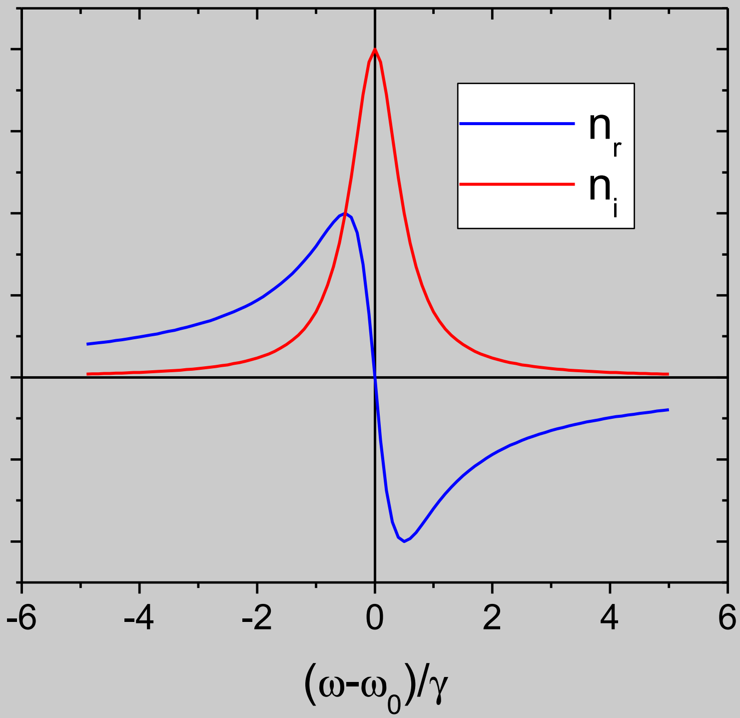
$$n^2(\omega) \approx 1 + \frac{N_a e^2}{\varepsilon_0 m_e} \frac{2\omega_0(\omega_0 - \omega) + i\omega_0\gamma}{4\omega_0{}^2(\omega_0 - \omega)^2 + \omega_0^2\gamma^2} = 1 + \frac{N_a e^2}{2\omega_0\varepsilon_0 m_e} \frac{(\omega_0 - \omega) + i\gamma/2}{(\omega_0 - \omega)^2 + (\gamma/2)^2}$$

# Complex refractive index near resonance

$$n(\omega) \approx \sqrt{1 + \frac{N_a e^2}{2\omega_0 \varepsilon_0 m_e} \frac{(\omega_0 - \omega) + i\gamma/2}{(\omega_0 - \omega)^2 + (\gamma/2)^2}}$$

- For low atomic density (e.g. gas) $n \approx 1$

Normalized plot of n-1 and k versus ω−$\omega_0$




$$n(\omega) \approx 1 + \frac{N_a e^2}{4\omega_0 \varepsilon_0 m_e} \frac{(\omega_0 - \omega) + i\gamma/2}{(\omega_0 - \omega)^2 + (\gamma/2)^2}$$

$$n_I(\omega) \approx \frac{N_a e^2}{4\omega_0 \varepsilon_0 m_e} \frac{\gamma/2}{(\omega_0 - \omega)^2 + (\gamma/2)^2}$$

$$n_R(\omega) \approx 1 + \frac{N_a e^2}{4\omega_0 \varepsilon_0 m_e} \frac{\omega_0 - \omega}{(\omega_0 - \omega)^2 + (\gamma/2)^2}$$

Lorentzian: FWHM = γ

# Complex refractive index

- When the incident light is near resonance, both Re and Im parts of *n*(ω) are important.
  - What is the meaning of a complex refractive index?
  - For a plane wave propagating in the z-direction:

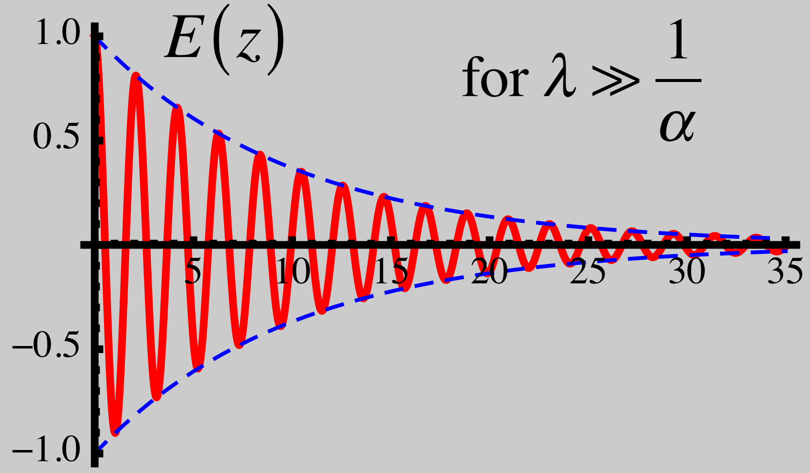
$$E(z,t) = E_0 e^{i(kz-\omega t)} = E_0 e^{i\left(\frac{\omega}{c}(n_R + i n_I)z - \omega t\right)} = E_0 e^{i\left(\frac{\omega}{c} n_R z - \omega t\right)} e^{-\frac{\omega}{c} n_I z}$$

For $n_I > 0$, absorption coefficient is

$$\alpha = \frac{\omega n_I}{2c}$$

For $n_I < 0$, gain coefficient is

$$g = \frac{\omega |n_I|}{2c}$$



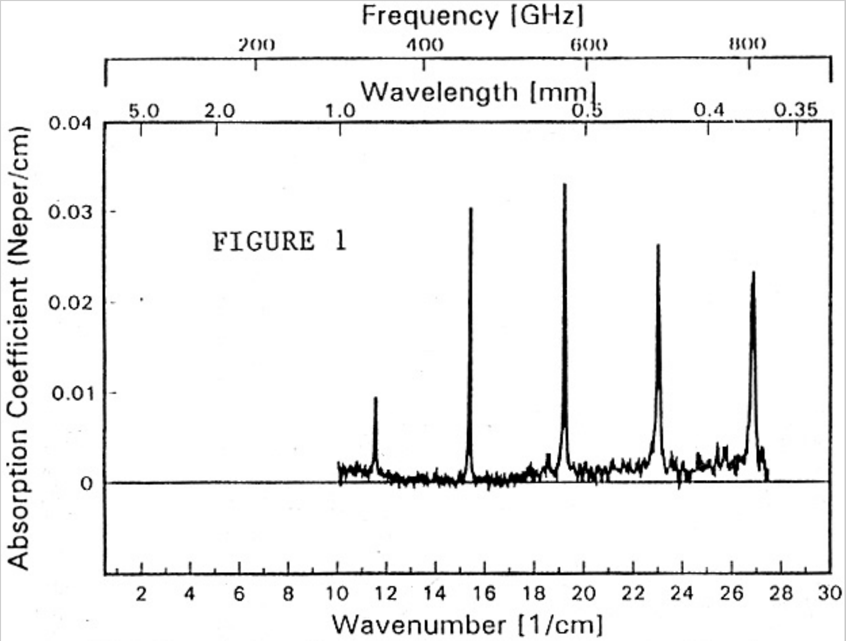
# refractive index for real gases

- In a real atom or molecule, there are many resonances

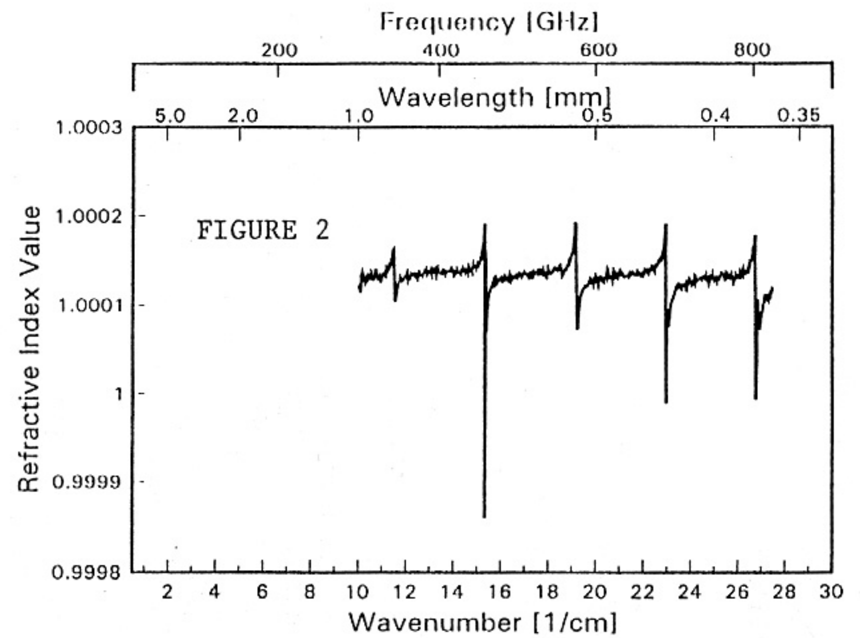
$$n^2 = 1 + \frac{N_a e^2}{\varepsilon_0 m_e} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 - i\omega\gamma_j)}$$

f = oscillator strength

$$\sum_j f_j = Z$$




*High Resolution Spectrum of Absorption Coeff. of Carbon Monoxide for 957mm path length(250 torr)*




*High Resolution spectrum of Refractive Index of Carbon Monoxide for 957mm path length(250 torr)*

http://www.ece.tufts.edu/research/mm-smm/facility/gases.html