2.3. **Fourier Series : Solution to the IVP.** Define,

$$
f(x) = \begin{cases} \dfrac{2k}{L}x, & 0 < x \le \frac{L}{2}, \\ \\ \dfrac{2k}{L}(L-x), & \frac{L}{2} < x < L. \end{cases} \tag{7}
$$

Let $L = 1$ and $k = 1$ and find the particular solution, which satisfies the initial displacement, $f(x)$, given by (7) and has zero initial velocity for all points on the object.

3. Inhomogeneous Wave Equation on a closed and bounded spatial domain of $\mathbb{R}^{1+1}$

Consider the non-homogeneous one-dimensional wave equation,

$$
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x,t) \quad , \tag{8}
$$

$$
x \in (0, L), \qquad t \in (0, \infty), \qquad c^2 = \frac{T}{\rho}, \tag{9}
$$

with boundary conditions and initial conditions,

$$
u(0,t) = u(L,t) = 0, \tag{10}
$$

$$
u(x,0) = u_t(x,0) = 0. \tag{11}
$$

Letting $F(x,t) = A\sin(\omega t)$ gives the following Fourier Series Representation of the forcing function F,

$$
F(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{L}\right), \tag{12}
$$

where

$$
f_n(t) = \frac{2A}{n\pi}(1 - (-1)^n)\sin(\omega t). \tag{13}
$$

3.1. **Educated Fourier Series Guessing.** Based on the boundary conditions we assume a Fourier sine series solution. However, the time-dependence is unclear. So, assume that,

$$
u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) G_n(t). \tag{14}
$$

where $G_n(t)$ represents the unknown dynamics of the n-th Fourier mode. Using this assumption and (12)-(13), show by direct substitution that (8) yields the ODE,

$$
\ddot{G}_n + \left(\frac{cn\pi}{L}\right)^2 G_n = \frac{2A}{n\pi}(1 - (-1)^n)\sin(\omega t). \tag{15}
$$

3.2. **Solving for the Dynamics.** The solution to (15) is given by,

$$
G_n(t) = G_n^h(t) + G_n^p(t), \tag{16}
$$

$\Rightarrow$

$$
u(x,t) = \underbrace{\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) G_n^h(t)}_{u_h(x,t) = \text{homogeneous sol'n}} + \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) G_n^p(t) \Big\} \text{Particular } u_p(x,t)
$$

where $G_n^h(t) = B_n \cos\left(\frac{cn\pi}{L}t\right) + B_n^* \sin\left(\frac{cn\pi}{L}t\right)$ is the homogeneous solution and $G_n^p(t)$ is the particular solution to (15).

3.2.1. *Particular Solution - I.* If $\omega \ne cn\pi/L$ then what would the choice for $G_n^p(t)$ be, assuming you were solving for $G_n^p(t)$ using the method of undetermined coefficients? DO NOT SOLVE FOR THESE COEFFICIENTS

3.2.2. *Particular Solution - II.* If $\omega = cn\pi/L$ then what would the choice for $G_n^p(t)$ be, assuming you were solving for $G_n^p(t)$ using the method of undetermined coefficients? DO NOT SOLVE FOR THESE COEFFICIENTS

3.2.3. *Physical Conclusions.* For the Particular Solution - II, what is $\lim_{t\to\infty} u(x,t)$ and what does this limit imply physically?

4. Vibrations of a Rectangular Membrane: Wave Equation on a Bounded Domain of $\mathbb{R}^{2+1}$

Suppose that you are given an infinitesimally thin, ideally elastic membrane of area $A = L_x L_y$, which is allowed to move in the z-axis direction but is permanently fixed along its perimeter. Use the solution to the corresponding PDE to describe the first four fundamental vibrational modes and the structure of their nodal lines.

9/19/12

Inhomogeneous PDE:

Recall: ~~If~~ Given $my'' + by' + ky = f(t)$ the general sol$\underline{n}$ is of the form

$$y(t) = y_h(t) + y_p(t)$$

where $y_h(t)$ is the sol$\underline{n}$ to

$$my'' + by' + ky = 0 \quad \Big\} \text{ Homogeneous Eq}\underline{n}.$$

We hope something similar occurs with

$$(\mathrm{I}) \qquad u_{tt} = c^2 u_{xx} + F(x,t), \quad x \in (0,L)$$

$$(\mathrm{II}) \qquad u(0,t) = 0, \; u(L,t) = 0 \qquad t \in (0,\infty)$$

$$(\mathrm{III}) \qquad u(x,0) = u_t(x,0) = 0$$

First, what is the $\overbrace{\text{homogeneous sol}\underline{n}}^{\text{for when } F(x,t)=0}$?

This will guide our sol$\underline{n}$ process.

Well, in that case we have a unforced string with fixed ends. The general sol$\underline{n}$ to this is given by

$$G_n^h(t) \equiv \text{Homogeneous dynamics}$$

$$U_h(x,t) = \underbrace{\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)}_{\text{Shape made by interference of spatial waves.}} \overbrace{\left[\begin{array}{l} A_n \cos(c\sqrt{\lambda_n}t) + \\ + B_n \sin(c\sqrt{\lambda_n}t) \end{array} \right]}^{G_n^h(t)}$$

We expect the shape $f\underline{n}$ will still be viable b/c of Fourier interference principles but the dynamics could be different.

Thus, we guess

$$(*) \quad U(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) G_n(t)$$

[Ack, I'm using G_n out of habit. You may want to use T_n]

Where $G_n(t)$ is an unknown dynamic.

Goal: Find $G_n(t)$!

How: Sub $(*)$ into nonhomogeneous PDE.

From (I) we have,

$$u_{tt} - c^2 u_{xx} - F(x,t) =$$

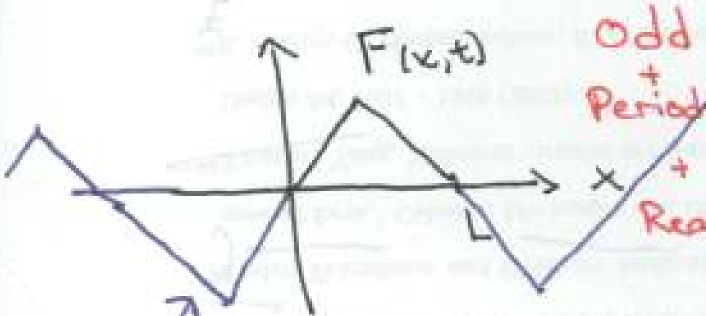
$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) G_n''(t) - c^2 \sum_{n=1}^{\infty} -\left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi}{L}x\right) G_n(t) - \underbrace{F(x,t)}_{*}$$

by linear ind. of sine f$\underline{n}$.

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) \overbrace{\left[G_n'' + \left(\frac{cn\pi}{L}\right)^2 G_n - \underbrace{f_n(t)}_{*} \right]}^{=0} = 0$$

Key Idea: We don't know what $F(x,t)$ is but it must be defined on only $(0,L)$. Thus, we can find an odd periodic Extension of F and therefore a Fourier Series Rep.

Graphically:



Odd + Periodic + Reasonable $\Rightarrow$

odd periodic wave $F^*(x,t)$.
s.t. $F^*(x,t) = F(x,t)$ for $x \in (0,L)$

Math:

$$F(x,t) = \sum_{n=1}^{\infty} f_n \underbrace{\sin\left(\frac{n\pi}{L}x\right)}_{\text{odd + periodic}}$$

$$\underbrace{f_n = \frac{2}{L}\int_0^L F(x,t)\sin\left(\frac{n\pi}{L}x\right)dx}_{\text{Reasonable}}$$

Thus for $u_{tt} - c^2 u_{xx} - F(x,t) = 0$ we Require

(*) $$G''_n + \underbrace{\left(\frac{cn\pi}{L}\right)^2}_{c^2\lambda_n,\ \sqrt{\lambda_n} = \frac{n\pi}{L}} G_n = f_n, \quad n = 1, 2, 3, \dots$$

Note:

- (*) $T''_n + c^2 \lambda_n T_n = f_n$ is the same time Eq$\underline{n}$ as before but now with an External forcing term, which results in new dynamics.

We are told to assume $f_n(t) = \frac{2A}{n\pi}\left(1 - (-1)^n\right)\sin(\omega t)$ thus,

$$G''_n + c^2 \lambda_n G_n = \frac{2A}{n\pi}\left(1 - (-1)^n\right)\sin(\omega t)$$

whose homogeneous Sol$\underline{n}$ is known as [Right?! From homogeneous fixed string.

$$G^h_n(t) = A_n \cos(c\sqrt{\lambda_n}t) + B_n \sin(c\sqrt{\lambda_n}t)$$

So, what is the particular sol$\underline{n}$ $G^p_n(t)$?

Well, that depends. Since $f_n(t) \propto \sin(\omega t)$ we should guess $G_n^p(t) = \alpha_n \sin(\omega t) + \beta_n \cos(\omega t)$.

That is, unless $\omega = \frac{cn\pi}{L}$, which means that $G_n^p(t)$ is the same as $G_n^h(t)$, up to constants. In that case

$$G_n^p(t) = \alpha_n t \sin\left(\frac{cn\pi}{L}t\right) + \beta_n t \cos\left(\frac{cn\pi}{L}t\right)$$

which has the unfourtunate behavior of

$$\lim_{t\to\infty} G_n^p(t) = \infty.$$

This is resonance of a forced string! Thus, if

$$\omega \neq \frac{cn\pi}{L} \Rightarrow u(x,t) = \sum_{n=1}^{\infty} \sin\Big(\overbrace{\frac{n\pi}{L}}^{\sqrt{\lambda_n}}x\Big) G_n(t) =$$

If $\omega = \frac{cn\pi}{L}$ in purple.

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) \underbrace{\begin{bmatrix} A_n \cos(c\sqrt{\lambda_n}t) + \\ + B_n \sin(c\sqrt{\lambda_n}t) \end{bmatrix}}_{G_n^h(t)} + \quad \Big\}\ \text{homogeneous Sol}_n$$

Particular Sol$_n$ $\Big\{$

$$+ \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) \underbrace{\left[\alpha_n^{t} \cos(c\sqrt{\lambda_n}t) + \beta_n^{t} \sin(c\sqrt{\lambda_n}t)\right]}_{G_n^p(t)}$$

If $\omega = cn\pi/L$ — If $\omega = cn\pi/L$

Via Undetermined coeff — Via Undetermined Coeff.