(a) What velocity-density curves are implied by the above general model, under the steady-state hypothesis?

(b) Consider the case proposed by Edie, $m = 1$, $l = 2$. Show that the assumption that $u(\rho_{max}) = 0$ does not yield a reasonable velocity-density curve. Instead, assume $u(0) = u_{max}$, and briefly discuss the resulting model.

64.4. Show that the two car-following models described in this section satisfy $dq/d\rho \leq 0$.

64.5. Compare the two theoretical models of this section to the Lincoln Tunnel data (see Sec. 62). Which theory best fits the data? [Hint: The parameters of the model are usually chosen by making a least-squares fit to the data. However, if you wish for simplicity you may assume that $\rho_{max} = 225$ cars/mile and also assume that the theoretical curve exactly satisfies the first data point, $u = 32$ m.p.h. when $\rho = 34$ cars/mile.]

65. Partial Differential Equations

For a given segment of a highway, experiments can be run to analyze the density dependence of the velocity. *If we assume that under all circumstances* the driver's velocity is a known function of ρ, determined by $u = u(\rho)$, then conservation of cars (equation 60.8) implies

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u(\rho)) = 0.} \tag{65.1}$$

This is a partial differential equation in one unknown variable ρ.

Suppose a nonconstant initial traffic density existed, as shown in Fig. 65-1. Different cars will move at different velocities (since the spacing is nonuniform). Thus the density will change immediately, and, under our assumptions, the drivers would adjust their velocities immediately. This process would continue. If we were interested in the density of cars at a later time we would "just" need to solve the partial differential equation.

The traffic problem has been formulated in terms of one partial differential equation, equation 65.1, or equivalently

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}q(\rho) = 0,} \tag{65.2}$$

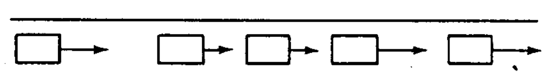

Figure 65-1 Nonuniform traffic.

since $q = \rho u$. q can be considered a function of ρ only. This last expression is often easier to use since by the chain rule

$$\frac{\partial}{\partial x}q(\rho) = \frac{dq}{d\rho}\frac{\partial \rho}{\partial x},$$

and thus

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial \rho}{\partial x} = 0.} \tag{65.3}$$

One appropriate condition in order to solve uniquely the partial differential equation is an initial condition. With an nth order ordinary differential equation, n initial conditions are needed. The number of conditions are correspondingly the same for partial differential equations. Thus for equation 65.3 only one initial condition is needed since the partial differential equation only involves one *time* derivative. However, there are some major differences between ordinary and partial differential equations due to the additional independent variable.

To illustrate this difference, let us consider three extremely simple first order partial differential equations:

(1) $\dfrac{\partial \rho}{\partial t} = 0$

(2) $\dfrac{\partial \rho}{\partial t} = -\rho + 2e^t$

(3) $\dfrac{\partial \rho}{\partial t} = -x\rho.$

These are called partial differential equations because ρ is assumed to depend on x and t (even though there is no explicit appearance of x in either of the first two equations). If ρ only depends on t, then the first two would be ordinary differential equations, the general solutions being:

(1) $\rho = \text{constant} = c_1$

(2) $\rho = c_2e^{-t} + e^t.$

To have a unique solution, one initial condition is needed, for example, if $\rho(0) = \rho_0$, then

(1) $\rho = \rho_0$

(2) $\rho = (\rho_0 - 1)e^{-t} + e^t.$

However, now we assume (as originally proposed) that ρ depends on both x and t.