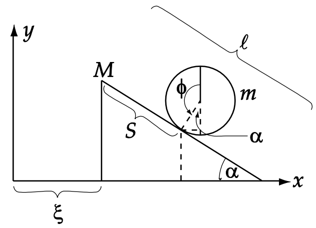
7-6.



Let us choose ξ,S as our generalized coordinates. The x,y coordinates of the center of the hoop are expressed by

$$\left.\begin{aligned} x &= \xi + S\cos\alpha + r\sin\alpha \\ y &= r\cos\alpha + (\ell - S)\sin\alpha \end{aligned}\right] \tag{1}$$

Therefore, the kinetic energy of the hoop is

$$\begin{aligned} T_{\text{hoop}} &= \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)+\frac{1}{2}I\dot{\phi}^2 \\ &= \frac{1}{2}m\left[\left(\dot{\xi}+\dot{S}\cos\alpha\right)^2+\left(-\dot{S}\sin\alpha\right)^2\right]+\frac{1}{2}I\dot{\phi}^2 \end{aligned} \tag{2}$$

Using $I = mr^2$ and $S = r\phi$, (2) becomes

$$T_{\text{hoop}} = \frac{1}{2}m\left[2\dot{S}^2+\dot{\xi}^2+2\dot{\xi}\dot{S}\cos\alpha\right] \tag{3}$$

In order to find the total kinetic energy, we need to add the kinetic energy of the translational motion of the plane along the x-axis which is

$$T_{\text{plane}} = \frac{1}{2}M\,\dot{\xi}^2 \tag{4}$$

Therefore, the total kinetic energy becomes

$$T = m\dot{S}^2+\frac{1}{2}(m+M)\,\dot{\xi}^2+m\dot{\xi}\dot{S}\cos\alpha \tag{5}$$

The potential energy is

$$U = mgy = mg\left[r\cos\alpha+(\ell-S)\sin\alpha\right] \tag{6}$$

Hence, the Lagrangian is

$$l = m\dot{S}^2+\frac{1}{2}(m+M)\,\dot{\xi}^2+m\dot{\xi}\dot{S}\cos\alpha - mg\left[r\cos\alpha+(\ell-S)\sin\alpha\right] \tag{7}$$

from which the Lagrange equations for ξ and S are easily found to be

$$\boxed{2m\ddot{S}+m\ddot{\xi}\cos\alpha - mg\sin\alpha = 0} \tag{8}$$

$$\boxed{(m+M)\,\ddot{\xi}+m\ddot{S}\cos\alpha = 0} \tag{9}$$

or, if we rewrite these equations in the form of uncoupled equations by substituting for $\ddot{\xi}$ and $\ddot{S}$, we have

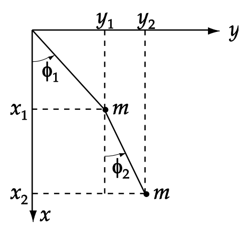
$$\left.\begin{aligned}\left[2-\frac{m\cos^2\alpha}{m+M}\right]\ddot{S}-g\sin\alpha=0\\ \ddot{\xi}=-\frac{mg\sin\alpha\cos\alpha}{2(m+M)-m\cos^2\alpha}\end{aligned}\right] \tag{10}$$

Now, we can rewrite (9) as

$$\frac{d}{dt}\left[(m+M)\dot{\xi}+m\dot{S}\cos\alpha\right]=0 \tag{11}$$

where we can interpret $(m+M)\dot{\xi}$ as the x component of the linear momentum of the total system and $m\dot{S}\cos\alpha$ as the x component of the linear momentum of the hoop with respect to the plane. Therefore, (11) means that the x component of the total linear momentum is a constant of motion. This is the expected result because no external force is applied along the x-axis.

7-7.



If we take (ϕ_1,ϕ_2) as our generalized coordinates, the x,y coordinates of the two masses are

$$\left.\begin{aligned}x_1&=\ell\cos\phi_1\\ y_1&=\ell\sin\phi_1\end{aligned}\right] \tag{1}$$

$$\left.\begin{aligned}x_2&=\ell\cos\phi_1+\ell\cos\phi_2\\ y_2&=\ell\sin\phi_1+\ell\sin\phi_2\end{aligned}\right] \tag{2}$$

Using (1) and (2), we find the kinetic energy of the system to be

$$\begin{aligned}T&=\frac{m}{2}\left(\dot{x}_1^2+\dot{y}_1^2\right)+\frac{m}{2}\left(\dot{x}_2^2+\dot{y}_2^2\right)\\ &=\frac{m}{2}\ell^2\left[\dot{\phi}_1^2+\dot{\phi}_1^2+\dot{\phi}_2^2+2\dot{\phi}_1\dot{\phi}_2\left(\sin\phi_1\sin\phi_2+\cos\phi_1\cos\phi_2\right)\right]\\ &=\frac{m}{2}\ell^2\left[2\dot{\phi}_1^2+\dot{\phi}_2^2+2\dot{\phi}_1\dot{\phi}_2\cos\left(\phi_1-\phi_2\right)\right] \end{aligned}\tag{3}$$

The potential energy is

$$U = -mgx_1 - mgx_2 = -mg\ell\left[2\cos\phi_1 + \cos\phi_2\right] \tag{4}$$

Therefore, the Lagrangian is

$$L = m\ell^2\left[\dot{\phi}_1^2 + \frac{1}{2}\dot{\phi}_2^2 + \dot{\phi}_1\dot{\phi}_2\cos\left(\phi_1 - \phi_2\right)\right] + mg\ell\left[2\cos\phi_1 + \cos\phi_2\right] \tag{5}$$

from which

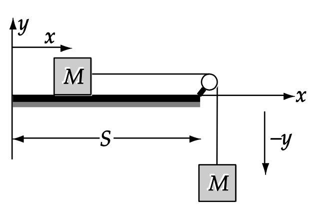
$$\left.\begin{aligned}
\frac{\partial L}{\partial \phi_1} &= m\ell^2\dot{\phi}_1\dot{\phi}_2\sin\left(\phi_1 - \phi_2\right) - 2mg\ell\sin\phi_1 \\
\frac{\partial L}{\partial \dot{\phi}_1} &= 2m\ell^2\dot{\phi}_1 + m\ell^2\dot{\phi}_2\cos\left(\phi_1 - \phi_2\right) \\
\frac{\partial L}{\partial \phi_2} &= -m\ell^2\dot{\phi}_1\dot{\phi}_2\sin\left(\phi_1 - \phi_2\right) - mg\ell\sin\phi_2 \\
\frac{\partial L}{\partial \dot{\phi}_2} &= m\ell^2\dot{\phi}_2 + m\ell^2\dot{\phi}_1\cos\left(\phi_1 - \phi_2\right)
\end{aligned}\right] \tag{6}$$

The Lagrange equations for ϕ_1 and ϕ_2 are

$$\boxed{2\phi_1 + \ddot{\phi}_2\cos\left(\phi_1 - \phi_2\right) + \dot{\phi}_2^2\sin\left(\phi_1 - \phi_2\right) + 2\frac{g}{\ell}\sin\phi_1 = 0} \tag{7}$$

$$\boxed{\ddot{\phi}_2 + \ddot{\phi}_1\cos\left(\phi_1 - \phi_2\right) - \dot{\phi}_1^2\sin\left(\phi_1 - \phi_2\right) + \frac{g}{\ell}\sin\phi_2 = 0} \tag{8}$$

7-10.



Let the length of the string be ℓ so that

$$\left(S - x\right) - y = \ell \tag{1}$$

Then,

$$\dot{x} = -\dot{y} \tag{2}$$

a) The Lagrangian of the system is

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M\dot{y}^2 - Mgy = M\dot{y}^2 - Mgy \tag{3}$$

Therefore, Lagrange's equation for y is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 2M\ddot{y} + Mg = 0 \tag{4}$$

from which

$$\ddot{y} = -\frac{g}{2} \tag{5}$$

Then, the general solution for y becomes

$$y(t) = -\frac{g}{4}t^2 + C_1 t + C_2 \tag{6}$$

If we assign the initial conditions $y(t=0)=0$ and $\dot{y}(t=0)=0$, we find

$$\boxed{y(t) = -\frac{g}{4}t^2} \tag{7}$$

b) If the string has a mass m, we must consider its kinetic energy and potential energy. These are

$$T_{\text{string}} = \frac{1}{2}m\dot{y}^2 \tag{8}$$

$$U_{\text{string}} = -\frac{m}{\ell}yg\frac{y}{2} = -\frac{mg}{2\ell}y^2 \tag{9}$$

Adding (8) and (9) to (3), the total Lagrangian becomes

$$L = M\dot{y}^2 - Mgy + \frac{1}{2}m\dot{y}^2 + \frac{mg}{2\ell}y^2 \tag{10}$$

Therefore, Lagrange's equation for y now becomes

$$(2M+m)\ddot{y} - \frac{mg}{\ell}y + Mg = 0 \tag{11}$$

In order to solve (11), we arrange this equation into the form

$$(2M+m)\ddot{y} = \frac{mg}{\ell}\left[y - \frac{M\ell}{m}\right] \tag{12}$$

Since $\frac{d^2}{dt^2}\left[y - \frac{M\ell}{m}\right] = \frac{d^2}{dt^2}y$, (12) is equivalent to

$$\frac{d^2}{dt^2}\left[y - \frac{M\ell}{m}\right] = \frac{mg}{\ell(2M+m)}\left[y - \frac{M\ell}{m}\right] \tag{13}$$

which is solved to give

$$y - \frac{M\ell}{m} = Ae^{\gamma t} + Be^{-\gamma t} \tag{14}$$

where

$$\gamma = \sqrt{\frac{mg}{\ell(2M+m)}} \tag{15}$$

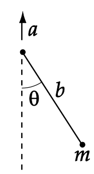
If we assign the initial condition $y(t=0)=0$; $\dot{y}(t=0)=0$, we have

$$A = +B = -\frac{M\ell}{2m}$$

Then,

$$\boxed{y(t) = \frac{M\ell}{m}\left(1-\cosh\gamma t\right)} \tag{16}$$

7-14.



$$x = b\sin\theta$$

$$y = \frac{1}{2}at^2 - b\cos\theta$$

$$\dot{x} = b\theta\cos\theta$$

$$\dot{y} = at + b\dot{\theta}\sin\theta$$

$$T = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) = \frac{1}{2}m\left(b^2\dot{\theta}^2 + a^2t^2 + 2abt\dot{\theta}\sin\theta\right)$$

$$U = mgy = mg\left[\frac{1}{2}at^2 - b\cos\theta\right]$$

$$L = T - U = \frac{1}{2}m\left(b^2\dot{\theta}^2 + a^2t^2 + 2abt\dot{\theta}\sin\theta\right) + mg\left(b\cos\theta - \frac{1}{2}at^2\right)$$

Lagrange's equation for θ gives

$$\frac{d}{dt}\left[mb^2\dot{\theta} + mabt\sin\theta\right] = mabt\dot{\theta}\cos\theta - mgb\sin\theta$$

$$b^2\ddot{\theta} + ab\sin\theta + abt\dot{\theta}\cos\theta = abt\dot{\theta}\cos\theta - gb\sin\theta$$

$$\boxed{\ddot{\theta} + \frac{a+g}{b}\sin\theta = 0}$$

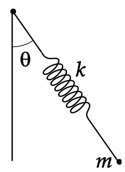
For small oscillations, $\sin\theta \simeq \theta$

$$\ddot{\theta}+\frac{a+g}{b}\theta=0\ .$$

Comparing with $\ddot{\theta}+\omega^2\theta=0$ gives

$$\boxed{T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{b}{a+g}}}$$

7-15.



b = unextended length of spring

ℓ = variable length of spring

$$T=\frac{1}{2}m\left(\dot{\ell}^2+\ell^2\,\dot{\theta}^2\right)$$

$$U=\frac{1}{2}k\left(\ell-b\right)^2+mgy=\frac{1}{2}k\left(\ell-b\right)^2-mg\,\ell\cos\theta$$

$$L=T-U=\frac{1}{2}m\left(\dot{\ell}^2+\ell^2\,\dot{\theta}^2\right)-\frac{1}{2}\left(\ell-b\right)^2+mg\,\ell\cos\theta$$

Taking Lagrange's equations for ℓ and θ gives

$$\ell:\frac{d}{dt}\left[m\dot{\ell}\right]=m\ell\dot{\theta}^2-k\left(\ell-b\right)+mg\cos\theta$$

$$\theta:\frac{d}{dt}\left[m\ell^2\dot{\theta}\right]=-mg\,\ell\sin\theta$$

This reduces to

$$\boxed{\begin{aligned}&\ddot{\ell}-\ell\dot{\theta}^2+\frac{k}{m}\left(\ell-b\right)-g\cos\theta=0\\&\ddot{\theta}+\frac{2}{\ell}\dot{\ell}\dot{\theta}+\frac{g}{\ell}\sin\theta=0\end{aligned}}$$

7-22. The potential energy U which gives the force

$$F(x,t)=\frac{k}{x^2}e^{-(t/\tau)} \qquad (1)$$

must satisfy the relation

$$F = -\frac{\partial U}{\partial x} \tag{2}$$

we find

$$U = \frac{k}{x} e^{-t/\tau} \tag{3}$$

Therefore, the Lagrangian is

$$\boxed{L = T - U = \frac{1}{2} m\dot{x}^2 - \frac{k}{x} e^{-t/\tau}} \tag{4}$$

The Hamiltonian is given by

$$H = p_x \dot{x} - L = \dot{x}\,\frac{\partial L}{\partial \dot{x}} - L \tag{5}$$

so that

$$\boxed{H = \frac{p_x^2}{2m} + \frac{k}{x} e^{-t/\tau}} \tag{6}$$

The Hamiltonian is equal to the total energy, $T + U$, because the potential does not depend on velocity, but the total energy of the system is not conserved because H contains the time explicitly.

7-23. The Hamiltonian function can be written as [see Eq. (7.153)]

$$H = \sum_j p_j \dot{q}_j - L \tag{1}$$

For a particle which moves freely in a conservative field with potential U, the Lagrangian in rectangular coordinates is

$$L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U$$

and the linear momentum components in rectangular coordinates are

$$\left.\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} \\ p_y &= m\dot{y} \\ p_z &= m\dot{z} \end{aligned}\right] \tag{2}$$

$$H = \left[m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2\right] - \left[\frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U\right]$$

$$= \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + U = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) \tag{3}$$

which is just the total energy of the particle. The canonical equations are [from Eqs. (7.160) and (7.161)]

$$\boxed{\begin{aligned} \dot{p}_x &= m\ddot{x} = -\frac{\partial U}{\partial x} = F_x \\ \dot{p}_y &= m\ddot{y} = -\frac{\partial U}{\partial y} = F_y \\ \dot{p}_z &= m\ddot{z} = -\frac{\partial U}{\partial z} = F_z \end{aligned}} \tag{4}$$

These are simply Newton's equations.