Lecture 1: Introduction and framing

Electromagnetism is a field theory. As such, it requires us to specify two things:

1) How matter produces fields.

2) How fields affect matter.

Note that for E+M, "matter" means charges.

Point (1) is handled by the Maxwell equations. These are our source equations, equations that describe how sources make fields.

$\nabla \cdot \vec{E} = \rho/\varepsilon_0$ — $\vec{E}$-fields with divergence come from charges (electric monopoles)

$\nabla \cdot \vec{B} = 0$ — $\vec{B}$-fields with divergence don't exist (no magnetic monopoles)

$\nabla \times \vec{E} = -\partial \vec{B}/\partial t$ — $\vec{E}$-fields with curl come from time-varying $\vec{B}$-fields - and those <u>only</u>

$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \left(\vec{J}_q + \vec{J}_D \right)$ — $\vec{B}$-fields with curl come from currents (moving electric monopoles) and from time-varying $\vec{E}$-fields. Sometimes we refer to $\varepsilon_0 \partial \vec{E}/\partial t$ as the displacement current and bundle the $\vec{J}$'s together.

group interpretation

Point (2) comes from the Lorentz force law: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Charges feel forces from $\vec{E}$-fields <u>always</u> and from $\vec{B}$-fields when the charges are in motion. This means that observers in different inertial reference frames might disagree on whether a particular charge is experiencing a magnetic force at all, but that's okay.

Eventually we discover that the fields themselves are different in different reference frames. This isn't a throwaway result since fields are very real things, carrying both energy + momentum.

There's another equation that often gets listed as fundamental - the so-called continuity equation: *

$$\nabla\cdot\vec{J} = -\partial\rho/\partial t$$

This is a statement of the local conservation of charge:
Not only is charge conserved globally, in that the total amount never changes, but it's also conserved locally, meaning that for charge to get from here to there, it has to move through the intervening space

Conservation of charge is certainly fundamental, but it isn't a postulate - we can derive it from other laws.

Take the divergence of both sides of the Ampere-Maxwell equation (which one is that?)

$$\nabla\cdot\left(\nabla\times\vec{B}\right) = \nabla\cdot\left(\mu_0\vec{J} + \mu_0\varepsilon_0\,\partial\vec{E}/\partial t\right)$$

$$0 = \mu_0\left[\nabla\cdot\vec{J} + \varepsilon_0\,\nabla\cdot\partial\vec{E}/\partial t\right]$$

$$0 = \nabla\cdot\vec{J} + \frac{\partial}{\partial t}\left(\varepsilon_0\,\nabla\cdot\vec{E}\right) \qquad \text{And} \quad \varepsilon_0\nabla\cdot\vec{E} = \rho \quad \text{(Gauss's Law)}$$

$$\Rightarrow \quad \nabla\cdot\vec{J} = -\partial\rho/\partial t$$

For charge to be entering or leaving a region, there must be a divergence in the current. Note that this gives another way to tell that Ampere's Law ($\nabla\cdot\vec{J} = \mu_0\vec{J}$) is incomplete.

(interlude: what are divergences & curls?
Clicker questions)

* There are many continuity equations for many contexts. "The" continuity equation is a bit of a misnomer.

These Maxwell equations are in differential form, which is arguably the cleanest form, describing what's happening at a single point in space, as compared to the integral forms that sample an entire region. We move back and forth between these forms using the divergence theorem & Stokes' theorem, which are higher dimensional generalizations of the fundamental theorem of calculus.

$\int_a^b f'(x)\,dx = f(b) - f(a)$ Info about function on 1-D domain contained in function's antiderivative at the domain's boundary (0-D points)

$\int (\nabla\times\vec{F})\cdot d\vec{A} = \oint \vec{F}\cdot d\vec{\ell}$ 2-D domain, 1-D boundary

$\int (\nabla\cdot\vec{F})\,dV = \oint \vec{F}\cdot d\vec{A}$ 3-D domain, 2-D boundary

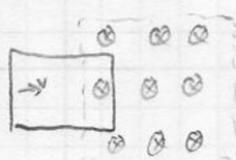
(group activity: convert Gauss's law from differential to integral & back using divergence theorem)

Correct integral expressions aren't always super transparent.

(Faraday clicker)

$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$ with no qualifiers is simply false and will lead you to conclude that curly E-fields exist when they don't

See, for example:



the equation is true when your boundary is not time-varying

$\oint \vec{E}\cdot d\vec{l} = -\int \frac{d\vec{B}}{dt}\cdot d\vec{A}$ is true always. And we can get the d/dt out using the 3D Leibniz rule (a kind of chain rule covering domains)

1D Leibniz: $\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = \frac{db}{dt} f(b,t) - \frac{da}{dt} f(a,t) + \int_a^b \frac{\partial}{\partial t} f(x,t)\,dx$

3D Leibniz: $\frac{d}{dt}\int_{A(t)} \vec{F}(\vec{x},t)\cdot d\vec{A} = -\oint(\vec{v}\times\vec{F})\cdot d\vec{\ell} + \int\left(\frac{\partial\vec{F}}{\partial t} + (\nabla\cdot\vec{F})\vec{v}\right)\cdot d\vec{A}$

$\vec{v} = \frac{d\vec{x}}{dt}$

Letting $\vec{B}$ be the vector field gives

$$\frac{d}{dt}\int_{A(t)} \vec{B}(\vec{x},t)\cdot d\vec{A} = \int \left(\frac{\partial \vec{B}}{\partial t} + \cancelto{0}{(\nabla\cdot\vec{B})\vec{v}} \right)\cdot d\vec{A} - \oint (\vec{v}\times\vec{B})\cdot d\vec{l}$$

$$\Rightarrow \quad \frac{d}{dt}\int \vec{B}(\vec{x},t)\cdot d\vec{A} + \oint (\vec{v}\times\vec{B})\cdot d\vec{l} = \int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

So if we sub into $\oint \vec{E}\cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$ we get

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A} - \oint (\vec{v}\times\vec{B})\cdot d\vec{l}$$

$$\Rightarrow \quad \oint (\vec{E}+\vec{v}\times\vec{B})\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$$

$$\mathcal{E}MF = -\frac{d\Phi}{dt}$$

Which is the correct integral form of Faraday's law, and explicitly shows E-field + motional contributions to the EMF.

Altogether, $\nabla\cdot\vec{E} = \rho/\varepsilon_0 \qquad \nabla\cdot\vec{B} = 0$

$\nabla\times\vec{E} = -\partial\vec{B}/\partial t \qquad \nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\,\partial\vec{E}/\partial t$

plus $\vec{F} = q(\vec{E}+\vec{v}\times\vec{B})$

Give us all of classical $E+M$. Everything else is sorting out details.