

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (5 Points) There is a population, P , of bacteria living in a Petri dish, which is known to increase at a rate proportional to the number of bacteria present at any time, t .

- (a) What is the differential equation that models this situation?

$$\frac{dP}{dt} = kP$$

- (b) Using your differential equation: If the population has doubled after 10 hours, then how long will it take to triple?

$$\begin{aligned} \int \frac{dP}{P} &= \int k dt \\ \ln P &= kt + C \\ P &= Ce^{kt} \\ P(0) &= C = P_0 \\ 2P_0 &= P_0 e^{10k} \\ 10k &= \ln 2 \\ k &= \frac{\ln 2}{10} \end{aligned}$$

$$\begin{aligned} 3P_0 &= P_0 e^{t \ln 2 / 10} \\ t \frac{\ln 2}{10} &= \ln 3 \\ t &= \frac{10 \ln 3}{\ln 2} \text{ hours} \end{aligned}$$

2. (10 Points) Find a general solution to the differential equation: explicitly

$$(1+t^2) \frac{dy}{dt} + 4ty = \frac{1}{(1+t^2)^2} \quad (1)$$

$$\begin{aligned} y' + \frac{4t}{1+t^2} y &= (1+t^2)^{-3} \\ u &= e^{\int \frac{4t dt}{1+t^2}} \quad u = 1+t^2 \\ &= e^{2 \ln(1+t^2)} \\ &= (1+t^2)^2 \\ y'(1+t^2)^2 + 4t(1+t^2)y &= \frac{1}{(1+t^2)^3} \\ \int (y(1+t^2)^2)' &= \int \frac{1}{(1+t^2)^3} \end{aligned}$$

$$y(1+t^2)^2 = \tan^{-1} t + C$$

$$y = \frac{\tan^{-1} t}{(1+t^2)^2} + \frac{C}{(1+t^2)^2}$$

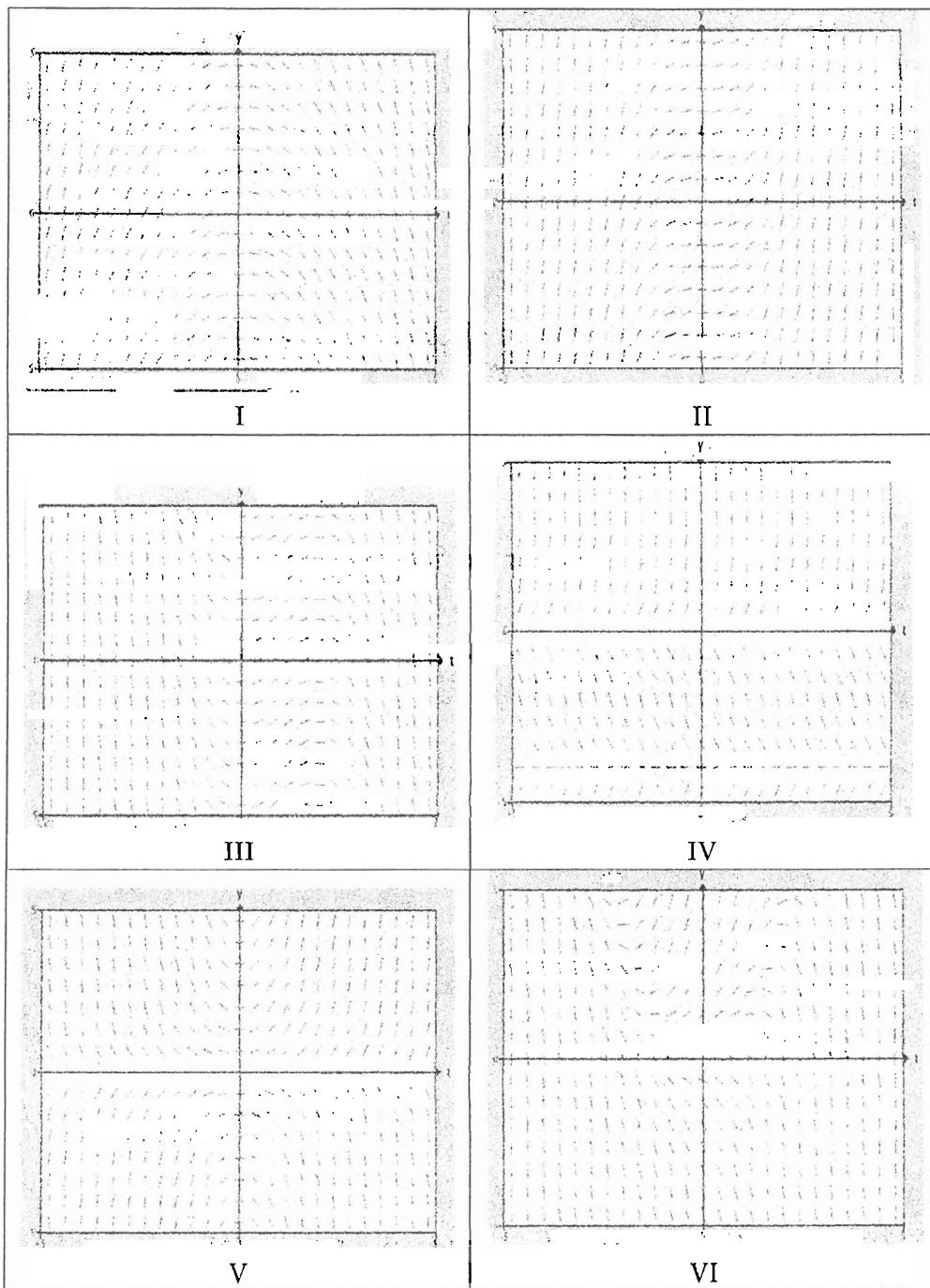
3. (8 Points) Match the following differential equations to their slope fields:

(a) $\frac{dy}{dt} = 2t - t^2$; III

(b) $\frac{dy}{dt} = y(y + 4)$, IV

(c) $\frac{dy}{dt} = yt$, V

(d) $\frac{dy}{dt} = y - t^2$, VI



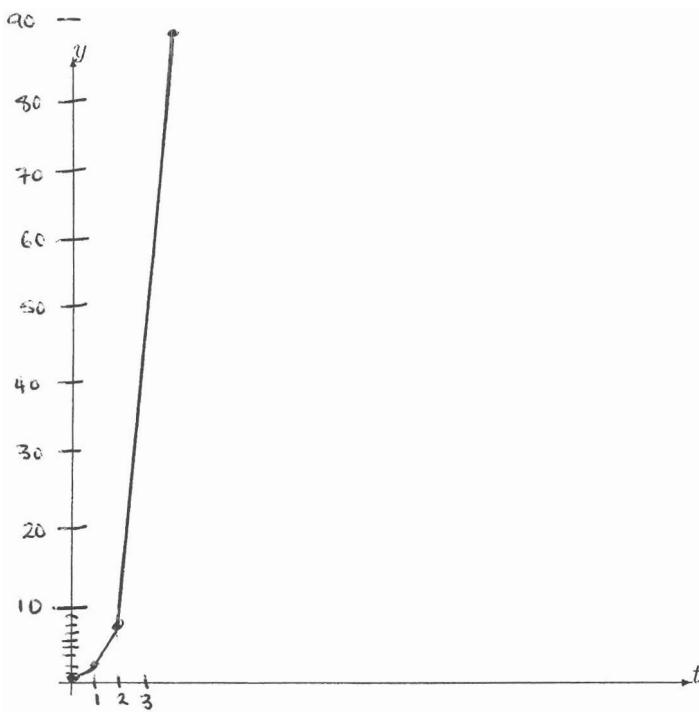
4. (8 Points) Consider the initial value problem,

$$\frac{dy}{dt} = y(y+t), \quad y(0) = +1. \quad (2)$$

- (a) Use Euler's Method with a step size of 1 on the interval $0 \leq t \leq 3$ to approximate the solution to the initial value problem.

k	t_k	y_k	$f(t_k, y_k)$
0	0	+1	1
1	1	2	6
2	2	8	80
3	3	88	

- (b) Graph the approximate solution on the following axis.



5. (5 Points) Given,

$$\frac{dy}{dt} = f(t, y). \quad (3)$$

Assume that f and $\frac{\partial f}{\partial y}$ are both continuous in the ty -plane and that $y_1(t) = 6 - t^2$ and $y_2(t) = -3 - t^2$ are solutions to the ODE. Suppose that $y(t)$ is a solution to the ODE such that $y(0) = 0$. What can you conclude about $y(t)$ for $t > 0$?

$y_1 + y_2$ are continuous $\forall t$

$$y_1(0) = 6, \quad y_2(0) = -3$$

$$-3 < 0 < 6$$

$$y_2(0) < y(0) < y_1(0)$$

3

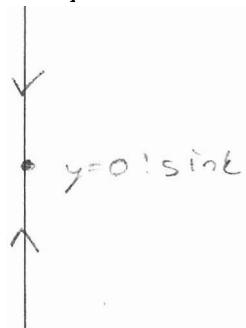
$$y_2(t) < y(t) < y_1(t)$$

$$-3 - t^2 < y(t) < 6 - t^2$$

6. (10 Points) Given, $\frac{dy}{dt} = e^{-y} - 1$.

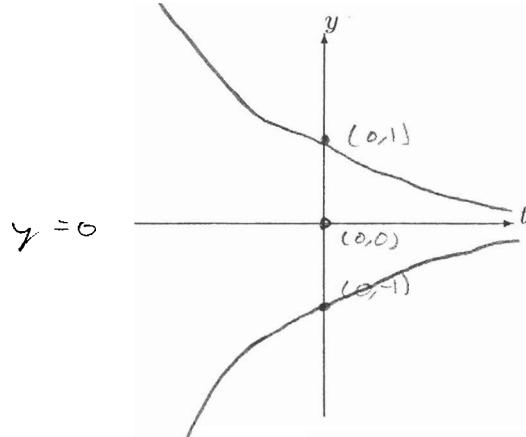
(a) Sketch the phase line and label the equilibrium points as sinks, sources or nodes.

$$\begin{aligned} f(y) &= e^{-y} - 1 = 0 \\ e^{-y} &= 1 \\ -y &= \ln(1) = 0 \\ y &= 0 \end{aligned}$$



$$\begin{aligned} f'(y) &= -e^{-y} \\ f'(0) &= -1 < 0 \text{ sink} \end{aligned}$$

(b) Sketch the graphs of the solutions satisfying the initial conditions $y(0) = 0$, $y(0) = 1$, and $y(0) = -1$.



7. (10 Points) Given, $\frac{dy}{dt} = ay - y^3$.

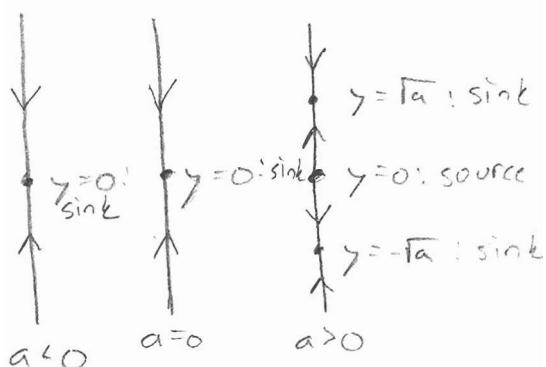
(a) Find the bifurcation value(s).

$$\begin{aligned} f(y) &= ay - y^3 \\ f'(y) &= a - 3y^2 = 0 \end{aligned}$$

$$\begin{aligned} \text{or EP: } f(y) &= y(a-y^2) = 0, \quad y=0, y=\pm\sqrt{a}, \quad \boxed{a=0} \\ a - 3y^2 &= 0 \\ 3y^2 &= a \\ y &= \pm\sqrt{\frac{a}{3}} \end{aligned}$$

$$\begin{aligned} f\left(\sqrt{\frac{a}{3}}\right) &= a\sqrt{\frac{a}{3}} - \frac{a}{3}\sqrt{\frac{a}{3}} = 0 \\ \boxed{a=0} \\ f\left(-\sqrt{\frac{a}{3}}\right) &= -a\sqrt{\frac{a}{3}} - \frac{a}{3}\sqrt{\frac{a}{3}} = 0 \\ \boxed{a=0} \end{aligned}$$

(b) Draw the phase lines for slightly smaller than, slightly larger than, and at the bifurcation value(s). Label your graphs and classify equilibrium points.



$$\begin{aligned} \text{EP: } f(y) &= y(a-y^2) = 0 \\ y=0, y &= \pm\sqrt{a} \end{aligned}$$

$a=0$: $y=0$! ! EP

$$\begin{aligned} f(y) &= -y^3 \\ y>0: f(y) &< 0 \\ y<0: f(y) &> 0 \end{aligned}$$

$a < 0$: $y=0, y \neq \pm\sqrt{a}$! ! EP

$$\begin{aligned} f(y) &= ay - y^3 \\ y>0: ay &- y^3 > 0 \\ y<0: ay &- y^3 > 0 \end{aligned}$$

$a > 0$: $y=0, y = \pm\sqrt{a}$: 3EP

$$f'(y) = a - 3y^2$$

$$f'(0) = a > 0, \text{ source}$$

$$f'(\sqrt{a}) = a - 3a = -2a < 0; \text{ sink}$$

$$f'(-\sqrt{a}) = a - 3a = -2a < 0, \text{ sink}$$

8. (6 Points) Solve the following differential equation via separation of variables. Express this solution explicitly.

$$\frac{dy}{dt} = (t+3)^2(y-1)$$

$$\int \left(\frac{1}{y-1}\right) dy = \int (t+3)^2 dt, \quad y \neq 1 \quad (\text{y}=1, \text{EP})$$

$$\ln|y-1| = \frac{1}{3}(t+3)^3 + C$$

$$|y-1| = e^C e^{\frac{1}{3}(t+3)^3}$$

$$y-1 = \pm e^C e^{\frac{1}{3}(t+3)^3}, \quad \boxed{y(t) = A e^{\frac{1}{3}(t+3)^3} + 1}, \quad \begin{cases} A=0, \pm e^C \\ \text{or } A=\pm e^C \end{cases}$$

9. (10 Points) Solve the following initial-value problem using the Method of Undetermined Coefficients.

$$\frac{dy}{dt} - y = 2\cos(3t), \quad y(0) = 2$$

$$\frac{dy_h}{dt} = y_h$$

$$y_h(t) = Kc^t$$

$$\frac{dy_p}{dt} - y_p = 2\cos(3t)$$

$$y_p(t) = \alpha \cos(3t) + \beta \sin(3t)$$

$$= -3\alpha \sin(3t) + 3\beta \cos(3t)$$

$$y(t) = Kc^t - \frac{1}{5}\cos(3t) + \frac{3}{5}\sin(3t)$$

$$y(0) = K - \frac{1}{5} = 2 \Rightarrow K = \frac{10}{5}$$

$$K = \frac{11}{5}$$

$$y(t) = \frac{11}{5}c^t - \frac{1}{5}\cos(3t) + \frac{3}{5}\sin(3t)$$

$$-3\alpha \sin(3t) + 3\beta \cos(3t) - (\alpha \cos(3t) + \beta \sin(3t)) = 2\cos(3t)$$

$$(-3\alpha - \beta) \sin(3t) + (3\beta - \alpha) \cos(3t) = 2\cos(3t)$$

$$\begin{aligned} -3\alpha - \beta &= 0 & 3\beta - \alpha &= 2 \\ \beta &= -3\alpha & -9\alpha - \alpha &= 2 \\ \beta &= \frac{3}{5}\alpha & -10\alpha &= 2 \\ & \alpha = -\frac{2}{10} = -\frac{1}{5} & \alpha &= -\frac{1}{5} \end{aligned}$$

$$y_p(t) = -\frac{1}{5}c^t \cos(3t) + \frac{3}{5}c^t \sin(3t)$$

10. (7 Points) Consider the following predator-prey model:

$$\frac{dx}{dt} = x - 0.5xy$$

$$\frac{dy}{dt} = -0.75y + 0.25xy$$

(a) Which variable represents the predator and which the prey? Justify your choice.

x : prey, Interaction term $(-\frac{1}{2}xy)$ negative y : predator, Interaction term $(\frac{1}{4}xy)$ positive

(b) Find all equilibrium points for the system.

$$\frac{dx}{dt} = x(1 - \frac{1}{2}y) = 0$$

$$x=0 \quad 1 - \frac{1}{2}y = 0$$

$$y=2$$

$$\frac{dy}{dt} = y(-\frac{3}{4} + \frac{1}{4}x) = 0$$

$$x=0 : y(-\frac{3}{4}) = 0$$

$$y=0$$

$$\boxed{\text{EP: } (0,0), (3,2)}$$

$$y=2 : 2(-\frac{3}{4} + \frac{1}{4}x) = 0$$

$$-\frac{3}{2} + \frac{1}{2}x = 0, x=3$$

(c) Modify the system to include the additional hunting of prey at a rate proportional to the number of prey. DO NOT SOLVE THIS NEW SYSTEM

$$\boxed{\begin{aligned} \frac{dx}{dt} &= x - 0.5xy - kx \\ \frac{dy}{dt} &= -0.75y + 0.25xy \end{aligned}}$$

11. (8 Points) Given the following differential equation,

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - x + x^3 = 0. \quad (8)$$

- (a) Convert the second-order ODE into a first-order system in terms of x and y where $y = \frac{dx}{dt}$.

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -3y + x - x^3$$

- (b) Find three constant solutions to the system of ODE's found in part (a).

$$\begin{aligned} x' &= y = 0 \\ y' &= -3y + x - x^3 = 0 \\ x - x^3 &= 0 \\ x(1-x^2) &= 0 \end{aligned}$$

$$\boxed{\begin{array}{l} (x,y) = (0,0) \\ (1,0) \\ (-1,0) \end{array}}$$

12. (13 Points) Consider the system:

$$\begin{aligned} \frac{dx}{dt} &= 2x + y^2 \\ \frac{dy}{dt} &= y \end{aligned}$$

- (a) Find the general solution of the system.

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= 2x + y^2 \\ y_h &= Ke^{2t} \quad \text{1pt} \\ y_p &= Ate^{-2t} \\ y &= Ke^{2t} + Ate^{-2t} \\ y &= C_1 e^{2t} + C_2 e^{-2t} \end{aligned}$$

$$\begin{aligned} x(t) &= Ke^{2t} + c^2 te^{-2t} \quad \text{1pt} \\ (y) &= \left(\begin{array}{c} Ke^{2t} + c^2 te^{-2t} \\ Ce^{-2t} \end{array} \right) \end{aligned}$$

- (b) From the general solution determine the unique solution passing through $(x(0), y(0)) = (6, 4)$.

$$\begin{pmatrix} x(0) = 6 \\ y(0) = 4 \end{pmatrix} \quad \begin{pmatrix} K + 0 \\ c \end{pmatrix}$$

$$K = 6, c = 4$$

$$x = (6e^{2t} + 16te^{-2t})$$

or integrating factor

$$\frac{dx}{dt} - 2x = c^2 e^{-2t}$$

$$(t) = e^{-2t}$$

$$x'e^{-2t} - 2xe^{-2t} = c^2$$

$$\{x e^{-2t}\} = \{c^2\}$$

$$x e^{-2t} = c^2 t + K$$

$$x = c^2 t e^{2t} + K e^{2t}$$