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(Meta)- Physics

Basic elements of physical theories:
1. Degrees of freedom - Things that can change
   - Propagating (test particles, small fluctuations, reacts to fixed backg)
   - Background (sources, background field config., current, etc.)
   - This split is largely for convenience.
2. Interactions - Why d.o.f. change

Example: E+M Maxwell $\rightarrow$ Sources create background fields
Lorentz $\rightarrow$ Background fields influence particles

Example: Gravity Ball dropped near earth surface,
Background is $\vec{g}$ from Earth and $\vec{F} = m\vec{g}$
gives influence on ball.

In an honest treatment, we should account for the "backreaction", i.e. the contribution to the background from the test particle and its influence on the original source. This is harder to calculate, but conceptually easier since everything is on the same footing.

We want a fundamental description, so let's avoid the split into background + prop. d.o.f. if possible!!

Okay so we have d.o.f. + interactions. What governs them?

1. You first studied classical particles and fields through equations of motion (axiomatic), e.g. $\vec{F} = m\vec{a}$, Maxwell, etc.
2. Then you learned about Lagrangians, Hamiltonians, Action principles which allow us to "derive" e.o.m. and handle more complicated situations (particularly w/ constraints since you focus on d.o.f.)
3. In non-relativistic QM (for particles) you learned that H is particularly useful.

At this point it seems that H is the most robust way to define how d.o.f. + interactions work.

But... enter relativity!

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H is conjugate to time, so not very space-time democratic. Lagrangians are more relativistically friendly.

Actually actions $\underbrace{S = \int L}$ = functional (takes a function and returns a #)

Note: e.o.m. can be expressed in terms of L, but deriving them requires S. Also, S includes topological information (which can modify e.o.m. from usual Euler-Lagrange equations)

Classically: $\frac{\delta S}{\delta(\text{d.o.f.})} = 0 \Rightarrow$ e.o.m. for d.o.f. Euler-Lagrange

Quantum : $M = M_1 + M_2 + \cdots$

$$\sum e^{i\frac{S}{\hbar}} = e^{i\frac{S_1}{\hbar}} + e^{i\frac{S_2}{\hbar}} + \cdots$$

over d.o.f.

Path Integral

So what does L look like? You have seen $L = T - U$ for $\vec{F} = -\vec{\nabla} U$

kinetic energy (T), potential energy (U)

However the language of potentials implies the background + prop. d.o.f. split which we want to avoid.

More generally: $L = \underbrace{\text{Kinetic}}_{\text{allows propagation of d.o.f.}} + \underbrace{\text{Interactions}}_{\text{encodes interactions between prop. d.o.f. w/ other prop. d.o.f. or backgrounds}}$

don't worry about this sign, we are using a diff. approach!

Relativistic vs. Non-relativistic

This really boils down to the symmetries we impose on L (or S actually).
NR - Galilean (Rotations, Translation in Space, "Translations" in Time)
R - Poincaré (Lorentz, Translations in space-time)

Also, in R the "rest-mass" energy gives an extra contribution to kinetic terms.

Point Particles vs. Fields

PP - d.o.f. are position $\vec{x}(t)$ (or $x^\mu(\tau)$) and internal states, e.g. $\chi^\pm$.

NR $S = \int L(\vec{x}(t), \dot{\vec{x}}(t), \chi)\,dt$ $\quad \frac{\delta S}{\delta \vec{x}} \Rightarrow$ e.o.m.
R $S = \int L(x^\mu(\tau), \partial_\tau x^\mu(\tau), \chi)\,d\tau$ $\quad \frac{\delta S}{\delta x^\mu} \Rightarrow$ e.o.m.

Fields - d.o.f. are field configurations $\phi(\vec{x},t)$ (or $\phi(x^\mu)$) w/ internal d.o.f. encoded in field value,

R $S = \int \mathcal{L}(\phi(x^\mu), \partial_\mu \phi(x^\mu))\,d^4x$ $\quad \frac{\delta S}{\delta \phi} = 0 \Rightarrow$ e.o.m. (KG, Dirac, Proca, Maxwell, etc.)

$\int D e^{i\frac{S}{\hbar}} \Rightarrow$ amplitudes (Feynman diagrams)

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$$S = \int L(x,\dot{x},t)\,dt \Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}}$$

$$S = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x \Rightarrow \boxed{\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = \frac{\partial \mathcal{L}}{\partial \phi}}$$

$\partial_\mu\phi$: $\frac{\partial \phi}{\partial x^\mu}$

If we want e.o.m. for non-interacting systems (free particles/fields) then:

Point Particle: $L = \frac{1}{2}m\dot{x}^2 \Rightarrow \frac{d}{dt}(m\dot{x}) = 0 \Rightarrow m\ddot{x} = 0$ Newton's 2nd Law

Scalar Field: $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi \Rightarrow \begin{cases} \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi \\ \frac{\partial\mathcal{L}}{\partial\phi} = -\left(\frac{mc}{\hbar}\right)^2\phi \end{cases} \Rightarrow \partial_\mu(\partial^\mu\phi) + \left(\frac{mc}{\hbar}\right)^2\phi = 0$

Klein-Gordon Equation

Spinor Field: $\mathcal{L} = i(\hbar c)\bar{\psi}\gamma^\mu\partial_\mu\psi - (mc^2)\bar{\psi}\psi$

$\bar{\psi}$: $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0$, $\frac{\partial\mathcal{L}}{\partial\bar{\psi}} = i\hbar c\gamma^\mu\partial_\mu\psi - mc^2\psi \Rightarrow (i\hbar c\gamma^\mu\partial_\mu - mc^2)\psi = 0$

ψ: $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = i\hbar c\bar{\psi}$, $\frac{\partial\mathcal{L}}{\partial\psi} = -mc^2\bar{\psi} \Rightarrow (i\hbar c\gamma^\mu\partial_\mu + mc^2)\bar{\psi} = 0$

Dirac Eq.

Vector Field: $\mathcal{L} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} + \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 A^\nu A_\nu \overset{w/\ m=0}{\Longrightarrow}$ Maxwell's Equations in Vacuum

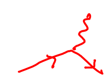
$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

So far we have looked at kinetic terms (with derivatives). What about interactions?

We know: $L = \frac{1}{2}m\dot{x}^2 - U(x) \Rightarrow m\ddot{x} = \underbrace{-\frac{\partial U}{\partial x}}_{\vec{F} = -\vec{\nabla}U}$ for particles.

For fields we don't want the split into background/test particle so no potential energies. Instead interactions will appear in $\mathcal{L}$ as products of field variables, e.g. $\bar{\psi}\psi A^\mu$

photon interacting w/ fermion



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So what interactions do we put in?

"The principle of local gauge invariance" also known as "Symmetries give rise to forces"

Recipe:
1. Observe a "global" symmetry of a non-interacting theory.
2. Promote this to a "local" or "gauge" symmetry.
3. Make sure everything can propagate.

Example: QED

$$\mathcal{L}_D = i\hbar c\,\bar\psi \overbrace{\gamma^\mu\partial_\mu}^{\not\partial}\psi - mc^2\bar\psi\psi$$

1. $\psi(x) \to e^{-i\underbrace{\frac{q}{\hbar c}}_{\text{bunch of constants}}\lambda}\psi(x) \Rightarrow \bar\psi(x)\to e^{i\frac{q}{\hbar c}\lambda}\bar\psi(x) \Rightarrow \mathcal{L}_D\to\mathcal{L}_D$

2. $\psi(x)\to e^{-i\frac{q}{\hbar c}\lambda(x)}\psi(x)$ local transformation $\Rightarrow \bar\psi(x)\to e^{i\frac{q}{\hbar c}\lambda(x)}\bar\psi(x) \Rightarrow \underbrace{\mathcal{L}_D\to\mathcal{L}_D + (q\bar\psi\gamma^\mu\psi)\partial_\mu\lambda(x)}_{\mathcal{L}_D \text{ is } \underline{\underline{\text{not}}} \text{ invariant unless } \lambda = \text{constant}}$

So we force it: $\mathcal{L}_D' = \mathcal{L}_D - (q\bar\psi\gamma^\mu\psi)A_\mu$ w/ $\psi\to e^{-i\frac{q}{\hbar c}\lambda(x)}\psi$
$\bar\psi\to e^{i\frac{q}{\hbar c}\lambda(x)}\bar\psi$
$A_\mu\to A_\mu+\partial_\mu\lambda(x)$

Then: $\mathcal{L}_D'\to\mathcal{L}_D + \cancel{(q\bar\psi\gamma^\mu\psi)}\partial_\mu\lambda(x) - (q\bar\psi\gamma^\mu\psi)A_\mu - \cancel{(q\bar\psi\gamma^\mu\psi)}\partial_\mu\lambda(x) = \mathcal{L}_D'$

Yes!!! But WTF!?

A_μ is simply the electromagnetic 4-vector potential (ϕ, A^i).

What we now have in $\mathcal{L}_D'$ is $\bar\psi\psi A_\mu$, i.e. an interaction!! $\Rightarrow$ E&M

3. So far A_μ can't propagate because it doesn't have its own kinetic term. We want to add one so consider:

$$\mathcal{L}_{proca} = -\frac{1}{16\pi}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 A^\nu A_\nu$$

However when $A_\mu\to A_\mu+\partial_\mu\lambda(x) \Rightarrow A^\nu A_\nu \to A^\nu A_\nu + 2(\partial^\nu\lambda(x))A_\nu + \partial^\nu\lambda\,\partial_\nu\lambda \neq A^\nu A_\nu$

So w/ $m\neq 0$ this kinetic term is not invariant. But that's okay because we know that $m_\gamma = 0$!

$$\mathcal{L}_{QED} = \mathcal{L}_D' + \mathcal{L}_{proca}(m=0) = \underbrace{i\hbar c\bar\psi\not\partial\psi - mc^2\bar\psi\psi - \frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}}_{\text{kinetic terms}} \; \underbrace{-q\bar\psi\gamma^\mu\psi A_\mu}_{\text{interaction}}$$

Another more elegant way to do this is:

$$\partial_\mu \to D_\mu \equiv \partial_\mu + i\frac{q}{\hbar c}A_\mu = \text{"covariant derivative"} \Rightarrow D_\mu\psi \to e^{-i\frac{q}{\hbar c}\lambda(x)}D_\mu\psi$$

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QCD

For QED $\psi \to \underbrace{e^{-i\frac{q}{\hbar c}\lambda}}_{U(1)\text{ transformation}}\psi$ $\quad U^\dagger U = \mathbb{I}_{1\times 1}$

Again start w/ $\mathcal{L}_D = i\hbar c\bar\psi\not\partial\psi - mc^2\bar\psi\psi$ but this time ψ includes a color state vector $c = \alpha\begin{pmatrix}1\\0\\0\end{pmatrix}+\beta\begin{pmatrix}0\\1\\0\end{pmatrix}+\gamma\begin{pmatrix}0\\0\\1\end{pmatrix}$.

1. For $\psi \to U\psi$ $\Rightarrow$ $\bar\psi \to \bar\psi U^\dagger$ $\Rightarrow$ $\mathcal{L}_D \to i\hbar c\bar\psi U^\dagger U\not\partial\psi - mc^2\bar\psi U^\dagger U\psi = \mathcal{L}_D$ if $\underbrace{U^\dagger U = \mathbb{I}_{3\times3}}_{U(3)}$

3x3 acting on color

A general SU(3) trans. $U = e^{-i\frac{q}{\hbar c}\overbrace{\lambda\cdot\phi}^{\lambda_1\phi_1+\lambda_2\phi_2+\cdots+\lambda_8\phi_8}}$

generators of SU(3) $\qquad$ parameters setting "amount" of each ind. transformation

For QCD we restrict to SU(3) since only 8 gluons exist.

Now we usually understand e^{matrix} by its Taylor expansion, but we must be very careful in this case since the generators do not commute $[\lambda_i, \lambda_j] = 2if_{ijk}\lambda_k$ (SU(3) is non-abelian)

2. We can make $\psi \to e^{-\frac{iq}{\hbar c}\lambda\cdot\phi(x)}\psi$ a good local symmetry w/ $\partial_\mu \to D_\mu \equiv \partial_\mu + i\frac{q}{\hbar c}\underbrace{\lambda\cdot A_\mu}_{\lambda_1 A_{1\mu}+\lambda_2 A_{2\mu}+\cdots}$

This requires 8 new gauge fields $A_{i\mu} \to A_{i\mu} + \partial_\mu\phi_i(x) + \frac{2q}{\hbar c}f_{ijk}\phi_j A_{k\mu}$

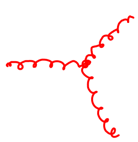
$\bar\psi\gamma^\mu D_\mu\psi$ now contains the interaction $\mathcal{L}_{int} = -(q\bar\psi\gamma^\mu\lambda\psi)\cdot A_\mu$

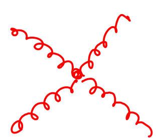
3. To allow $A_{i\mu}$ to propagate use $\mathcal{L}_{proca}$ w/ $m=0$ (for invariance):

$\mathcal{L}_{proca} = -\frac{1}{16\pi}F_i^{\mu\nu}F_{i\mu\nu}$ but now we need $F_i^{\mu\nu} = \partial^\mu A_i^\nu - \partial^\nu A_i^\mu - \frac{2q}{\hbar c}f_{ijk}A_j^\mu A_k^\nu$

which leads to (from $F^{\mu\nu}F_{\mu\nu}$):

$A^\mu A^\nu\partial_\mu A_\nu$ $\qquad\qquad$ $A^\mu A^\nu A_\mu A_\nu$





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Weak Interactions

We can play the same game for the SU(2) symmetry adding 3 fields $W^{\pm}, Z^0$ to get a local gauge symmetry.

But... to let these propagate we need a kinetic term for spin-1 fields ($\mathcal{L}_{proca}$) but now w/ $m \neq 0$!

Recall that $\mathcal{L}_{proca}$ is <u>not</u> invariant when $m \neq 0$, so are we screwed? No said Mr. Peter Higgs!! And the LHC backed him up!

In the end:

$$\mathcal{L} = \underbrace{i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - \frac{1}{16\pi}\overset{\partial_\mu A_\nu - \partial_\nu A_\mu}{\overset{\downarrow}{F^{\mu\nu}}}F_{\mu\nu}}_{\text{Kinetic Terms}} \; \underbrace{-q\bar{\psi}\gamma^\mu\psi A_\mu}_{\text{Interaction}}$$

This is the Lagrangian of QED which describes the interactions of fermions ψ with photons A_μ and includes Maxwell's equations!

Another way to summarize what we did is:

Start with a non-interacting theory $\mathcal{L}_{Dirac}$, then:
- Replace $\partial_\mu \to D_\mu \equiv \partial_\mu + iqA_\mu$ (a locally gauge <u>covariant</u> derivative)
- Add a gauge kinetic term $F^{\mu\nu}F_{\mu\nu}$

Does this sound anything like GR?
We started w/ flat space with no gravity (interactions).
To add gravity we had to:
- Replace $\partial_\mu \to \nabla_\mu \simeq \partial_\mu + \Gamma_\mu^{\bullet}$.
- Introduce $R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$ and add R to the Lagrangian.

So what is the global symmetry that is made local in GR? Lorentz Invariance! There is still an open debate as to whether GR is a gauge theory at the deepest level. But I still find the parallels striking. Even more so if we consider ...

Fiber Bundles

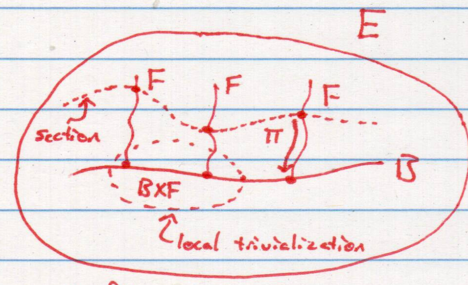
A fiber bundle E consists of:

- A base space B
- Fibers F

which locally resembles $B \times F$

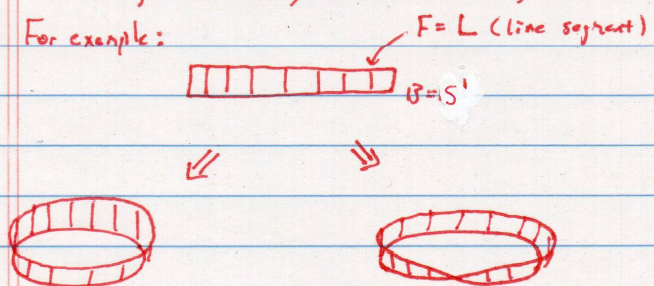
with a projection map $\pi : E \to B$

A section of the bundle is a choice of a point on each fiber.



Even though it locally looks like $B \times F$, it doesn't have to globally!

For example:



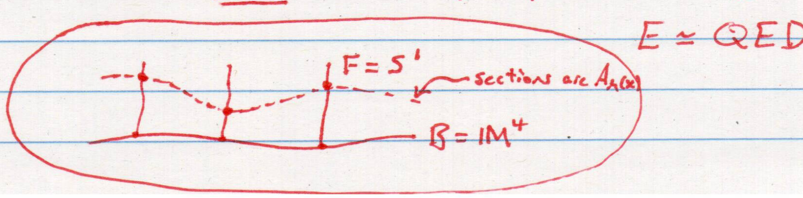
Globally $S^1 \times L$ = Cylinder   Globally $S^1 \times L_{\updownarrow}$ = Mobius Strip

Both cases locally look like $S^1 \times L$!

Now let's go back to QED for a moment. Recall the transformation

$\psi \to \psi' = \underbrace{e^{-iq\lambda(x)}}\psi$

This complex number of unit magnitude represents "rotations" around the unit circle in the complex plane. This suggests:

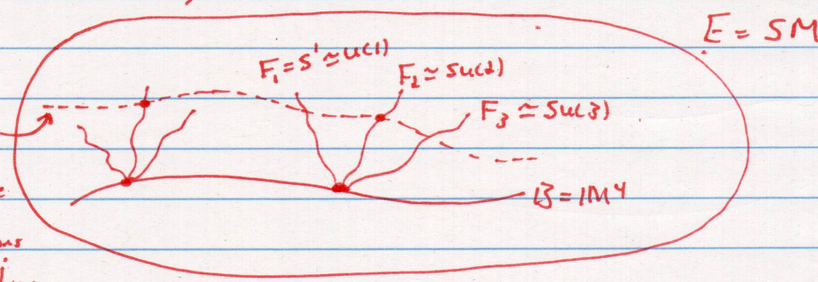


The interactions between ψ and $A_\mu(x)$ arise because we have to define a derivative operator which is good not just on the base (as ∂_μ would be) but rather on the entire bundle structure (which requires D_μ).

The section of the bundle $A_\mu(x)$ carries one spacetime index (and will for any bundle fiber) but also secretly carries gauge indices since we will generally have matrix-valued gauge symmetry transformations. In this case the "matrix" is a 1×1, so we don't really need the indices. Otherwise we would write $A_\mu^{ij}(x)$ for more general sections.

Now if the fibers in the bundle form vectors spaces then we have a "vector bundle". In fact all of the Standard Model forces can be viewed this way:

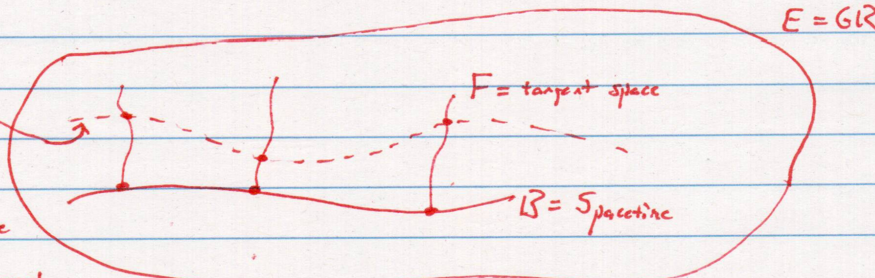
In this case, the sections have one spacetime vector index μ, and 2 gauge indices since the transformations require matrices, e.g. $A_\mu^{ij}(x)$



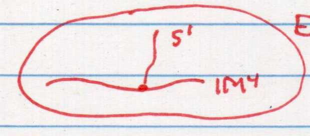
What about GR? Well the big difference between this picture and GR is that the local transformations of GR (Lorentz transformations) act in spacetime itself rather than in some abstract vector space of gauge degrees of freedom ($U(1)$, $SU(2)$, etc.).

So does GR have this bundle structure? Yes it does! Recall that we have a natural vector space in GR that is "fibered" over spacetime, i.e. attached at each point. That's the tangent space! But the tangent space is actually where we defined vectors (and tensors) to live, so it is natural to say that local Lorentz transformations act on the tangent (and cotangent) bundles:

In this case we get the usual spacetime μ index for sections, but we get 2 more from the tangent space, e.g. $\Gamma^\mu_{\nu\lambda}(x)$

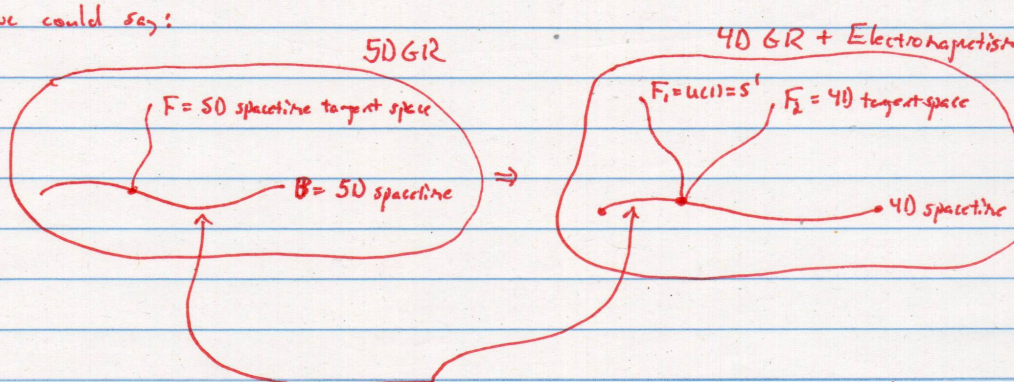


Now we should be careful because something like:



is **not** a 5D spacetime. If it where we would have to allow spacetime transformations which "mix" $\mathbb{M}^4$ and S^1. Indeed if we did GR on this space we would have a 5D tangent space and a 5D base space.

However, if the "mixing" was somehow forbidden, then it seems like we could say:



How do we do this? By making S^1 so small that nothing can move along it.

This is the Kaluza-Klein mechanism for getting 4D GR + E&M from pure 5D GR!