MATH348-Advanced Engineering Mathematics		Homework: PDEs - Part III
Wave Equations: Traveling and Standing Waves, Nodal Lines, and Nonlinear Equations		
Text: 12.2, 12.8	Lecture Notes : N/A	Lecture Slides: N/A

Quote of Homework Six
Our vibrations were getting nasty. But why? Was there no communication in this car? Had we deteriorated to the level of dumb beasts?
Duke : Fear and Loathing in Las Vegas (1998)

1. D'alembert Solution to the Wave Equation in $\mathbb{R}^{1+1}$

Show that by direct substitution the function $u(x,t)$ given by,

$$u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(y)dy, \tag{1}$$

is a solution to the one-dimensional wave equation where u_0 and v_0 are the ideally elastic objects initial displacement and velocity, respectively. [1]

2. Wave Equation on a closed and bounded spatial domain of $\mathbb{R}^{1+1}$

Consider the one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad , \tag{2}$$

$$x \in (0,L)\,, \qquad t \in (0,\infty)\,, \qquad c^2 = \frac{T}{\rho}. \tag{3}$$

Equations (2)-(3) model the time-evolution of the displacement from rest, $u = u(x,t)$, of an elastic medium in one-dimension. The object, of length L, is assumed to have a homogeneous lateral tension T, and linear density ρ. That is, $T, \rho \in \mathbb{R}^+$. Assume, as well, the boundary conditions[2],

$$u_x(0,t) = 0, u_x(L,t) = 0, \tag{4}$$

and initial conditions,

$$u(x,0) = f(x), \tag{5}$$

$$u_t(x,0) = g(x). \tag{6}$$

2.1. **Separation of Variables : General Solution.** Assume that the solution to (2)-(3) is such that $u(x,t) = F(x)G(t)$ and use separation of variables to find the general solution to (2)-(3), which satisfies (4)-(6). [3] [4]

2.2. **Qualitative Dynamics.** Describe how the the general solution to (2)-(3) changes as the tension, T, is increased while all other parameters are held constant. Also, describe how the solution changes when the linear density, ρ, is increased while all other parameters are held constant.

[1]This is called the d'Alembert solution to the wave equation. To do this you may want to recall the fundamental theorem of calculus, $\frac{d}{dx}\int_0^x f(t)dt = f(x)$ and properties of integrals, $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$.

[2]These boundary conditions imply that the object must have zero slope at each endpoint.

[3]It is important to notice that the solution to the spatial portion of the problem is the same as the heat problem above.

[4]Remember that in this case we have a nontrivial spatial solution for zero eigenvalue. From this you should find the associated temporal function should find that $G_0(t) = C_1 + C_2 t$.

2.3. **Fourier Series : Solution to the IVP.** Define,

$$f(x) = \begin{cases} \dfrac{2k}{L}x, & 0 < x \leq \frac{L}{2}, \\ \dfrac{2k}{L}(L-x), & \frac{L}{2} < x < L. \end{cases} \tag{7}$$

Let $L = 1$ and $k = 1$ and find the particular solution, which satisfies the initial displacement, $f(x)$, given by (7) and has zero initial velocity for all points on the object.

3. Inhomogeneous Wave Equation on a closed and bounded spatial domain of $\mathbb{R}^{1+1}$

Consider the non-homogeneous one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x,t) \quad , \tag{8}$$

$$x \in (0, L)\,, \qquad t \in (0, \infty)\,, \qquad c^2 = \frac{T}{\rho}. \tag{9}$$

with boundary conditions and initial conditions,

$$u(0,t) = u(L,t) = 0, \tag{10}$$

$$u(x,0) = u_t(x,0) = 0. \tag{11}$$

Letting $F(x,t) = A\sin(\omega t)$ gives the following Fourier Series Representation of the forcing function F ,

$$F(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{L}\right), \tag{12}$$

where

$$f_n(t) = \frac{2A}{n\pi}(1 - (-1)^n)\sin(\omega t). \tag{13}$$

3.1. **Educated Fourier Series Guessing.** Based on the boundary conditions we assume a Fourier sine series solution. However, the time-dependence is unclear. So, assume that,

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) G_n(t), \tag{14}$$

where $G_n(t)$ represents the unknown dynamics of the n-th Fourier mode. Using this assumption and (12)-(13), show by direct substitution that (8) yields the ODE,

$$\ddot{G}_n + \left(\frac{cn\pi}{L}\right)^2 G_n = \frac{2A}{n\pi}\left(1 - (-1)^n\right)\sin(\omega t). \tag{15}$$

3.2. **Solving for the Dynamics.** The solution to (15) is given by,

$$G_n(t) = G_n^h(t) + G_n^p(t), \tag{16}$$

where $G_n^h(t) = B_n \cos\left(\frac{cn\pi}{L}t\right) + B_n^* \sin\left(\frac{cn\pi}{L}t\right)$ is the homogeneous solution and $G_n^p(t)$ is the particular solution to (15).

3.2.1. *Particular Solution - I.* If $\omega \neq cn\pi/L$ then what would the choice for $G_n^p(t)$ be, assuming you were solving for $G_n^p(t)$ using the method of undetermined coefficients? DO NOT SOLVE FOR THESE COEFFICIENTS

3.2.2. *Particular Solution - II.* If $\omega = cn\pi/L$ then what would the choice for $G_n^p(t)$ be, assuming you were solving for $G_n^p(t)$ using the method of undetermined coefficients? DO NOT SOLVE FOR THESE COEFFICIENTS

3.2.3. *Physical Conclusions.* For the Particular Solution - II, what is $\lim_{t\to\infty} u(x,t)$ and what does this limit imply physically?

4. Vibrations of a Rectangular Membrane: Wave Equation on a Bounded Domain of $\mathbb{R}^{2+1}$

Suppose that you are given an infinitesimally thin, ideally elastic membrane of area $A = L_x L_y$, which is allowed to move in the $z-$axis direction but is permanently fixed along its perimeter. Use the solution to the corresponding PDE to describe the first four fundamental vibrational modes and the structure of their nodal lines.

5. Aspects of Nonlinearity

We have studied both ideal/linear waves and diffusive flows. Waves tend to oscillate and/or transport initial data while diffusive flows tend to smooth and spread this data out. In this way they are archetypes of more complicated flows and vibrations found in nature. However, these more complicated flows are often modeled by nonlinear differential equations whose mathematical construct is complicated in incomplete.[5] In the following we discuss some nonlinear PDE and their relation to physical phenomenon.

5.1. **Nonlinear Heat/Diffusion Equations.** Recall that the heat/diffusion equation was derived from $u_t + \text{div}(\boldsymbol{\phi}) = 0$ where $\boldsymbol{\phi} = -D\text{grad}(u)$. This physical parameter D was called the diffusivity and we assumed it to be constant throughout the medium. Generally, however, it makes sense to assume that $D = D(u, x, y, z, t)$, which lets the diffusivity depend on space-time and, most importantly, the unknown function, u. When this is the case the problem is said to be nonlinear and has found application in population modeling, fluid filtration and image processing. Consider `http://en.wikipedia.org/wiki/Anisotropic_diffusion` and the PeronaMalik diffusion videos on `http://www.youtube.com/user/arclnx?blend=4&ob=5#p/u/3/KIA4feoyxFY`. What is the choice of diffusivity coefficient and applications of nonlinear diffusion.

5.2. **Nonlinear Ocean Waves.** The study of rouge waves, `http://en.wikipedia.org/wiki/Rogue_wave`, which are fairly isolated and extremely large waves on the ocean surface, is still a matter of active research. Since they are difficulty to reproduce experimentally, much of the work is on mathematical modeling of such waves. Consider `http://en.wikipedia.org/wiki/Rogue_wave#Causes` and list the three nonlinear equations that are mentioned and when these equations are applicable to ocean wave modeling.

5.3. **Nonlinear Wave Equations.** For small kinetic energies waves, like electromagnetic or gravitational, travel according to the linear wave equation. As kinetic energies become larger, Einstein's equation becomes nonlinear and nonlinear waves are predicted. While this is a common entry point of nonlinear wave equations, there are nonlinear wave equations modeling phenomenon on terra firma. For example, consider `http://en.wikipedia.org/wiki/Wave_equation`, `http://en.wikipedia.org/wiki/Mach_number` and `http://en.wikipedia.org/wiki/Shock_wave`, what is a shock wave, how does this relate to mach number and how does one change the linear wave equation to model such phenomenon?

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[5] On reason nonlinear PDE are so difficult is that superposition, which was what gave rise to our Fourier series, does not generally hold. Actually, one can show that for many nonlinear problems the energies associated with Fourier modes is permitted to transfer between high and low frequency states. This can lead to pretty difficulties where problems starting off as 'energetically reasonable' do not stay so for all time. Yikes!