Exam 12/2

Note Title 11/27/2006

4 questions

Fourier transform

power series solution

sep. of variables

Laplace Equation

HW 8 out today

will put review notes up on web.

11/27/06

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$$\nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t} = 0$$

$$f(r, \theta, t)$$

Fourier Transform $t \to \omega$

$$\frac{\partial^2}{\partial t^2} \to (i\omega)^2 = -\omega^2$$

$$\frac{-\omega^2}{-c^2} = +k^2$$

$$\nabla^2 f + k^2 f = 0$$

$$f(r, \theta, \omega)$$

FT wave equation

Helmholtz eqn

We will solve for a single frequency. So

$$f = f(r, \theta)$$

Guess:

$$f(r,\theta) = R(r)\,P(\theta)$$

$\nabla^2 f$ in cylindrical coord.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

$$\Rightarrow \boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + k^2 f = 0}$$

Helmholtz eqn in cylindrical coord.

$f = RP$

$$\frac{P}{r}\frac{d}{dr}(rR') + \frac{1}{r^2}RP'' + k^2 RP = 0$$

divide by RP

$$\frac{1}{rR}\frac{d}{dr}(rR') + \frac{1}{r^2}\frac{P''}{P} + k^2 = 0$$

$\times\ r^2$:

$$\frac{r}{R}\frac{d}{dr}(rR') + k^2r^2 = -\frac{P''}{P} = \alpha^2$$

$$P''(\theta) + \alpha^2 P(\theta) = 0$$

$$P = P_0 e^{i\alpha\theta}$$

α must be integer

why?

leaves

$$\frac{r}{R}\frac{d}{dr}(rR') + k^2r^2 - n^2 = 0$$

$$r\frac{d}{dr}(rR') + (k^2r^2 - N^2)R = 0$$

$$\boxed{r^2 R'' + rR' + (k^2r^2 - N^2)R = 0}$$

change to dimensionless variable $kr = z$

$$\frac{1}{k}\frac{d}{dr} = \frac{d}{dz}$$

$$\frac{k^2r^2}{k^2}\frac{d^2}{dr^2}R + \frac{kr}{k}\frac{dR}{dr} + (k^2r^2 - N^2)R = 0$$

$$z^2\frac{d^2}{dz^2}R + z\frac{d}{dz}R + (z^2 - N^2)R = 0$$

$$z^2 R''(z) + zR'(z) + (z^2 - N^2)R(z) = 0$$

Bessel's equation

See equation 3 in mathworld page on <u>Bessel Function</u>

The solutions of which are finite at the origin are Bessel functions of the first kind

$$J_n(kr) = J_n(z)$$

Remember $k = \frac{\omega}{c}$

BesselJ vs BesselY in mathematica

Solutions of the 2D cylindrical Helmholtz equation

$$\sum_{n=0}^{\infty} A_n J_n(kr)\, e^{in\theta}$$

modes are

$$J_n(kr)\, e^{in\theta}$$

$n=0$ $\qquad J_0(kr)$

$n=1$ $\qquad J_1(kr)\, e^{i\theta}$

J vs Y

$\nabla^2 \phi = 0$ is spherical

$$\left(A_{\ell m} r^{\ell} + B_{\ell m} r^{-(\ell+1)}\right) Y_{\ell m}$$