Anisotropic media: birefringence.

in an isotropic medium:

$$\vec{D} = \varepsilon \vec{E} \qquad\qquad \vec{D} = \varepsilon_0 \varepsilon \vec{E}$$

if linear, ε doesn't depend on $\vec{E}$

$\vec{D} \parallel \vec{E}$

in an anisotropic medium, $\vec{D} \nparallel \vec{E}$

$$\underset{\substack{\uparrow \\ \text{applied}}}{\vec{D} = \vec{E}} + \underset{\substack{\nwarrow \\ \text{induced polarization}}}{4\pi \vec{P}} \qquad \boxed{\vec{D} = \varepsilon_0 \vec{E} + \vec{P}}$$

b/c of asymmetry of crystal, $\vec{P} \nparallel \vec{E}$ e.g.

write $\vec{D} = \overleftrightarrow{\varepsilon} \cdot \vec{E}$

$\overleftrightarrow{\varepsilon}$ = dielectric tensor = $\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}$

Guenther p535
if material is not lossy, energy considerations →
$\overleftrightarrow{\varepsilon}$ is Hermitian $\varepsilon_{ij} = \varepsilon_{ji}^*$

∴ We can choose the orientation of coordinates to diagonalize (in most crystals)

$$\rightarrow \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \varepsilon_x & & \\ & \varepsilon_y & \\ & & \varepsilon_z \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

this coord. system is aligned along the crystal axes.

so if $\vec{E} = E_x(x,y,z)\hat{x}$,

$\vec{D} = D_x \hat{x} = \varepsilon_x E_x$ only

biaxial crystal: $\varepsilon_x \neq \varepsilon_y \neq \varepsilon_z$

uniaxial crystal: one pair is the same. e.g. $\varepsilon_x = \varepsilon_y \neq \varepsilon_z$

isotropic crystal $\varepsilon_x = \varepsilon_y = \varepsilon_z = \varepsilon \qquad \vec{D} = \varepsilon \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \vec{E}$

"dichroic" absorption depends on polarization e.g. polaroid.

Wave propagation in birefringent media.

main points:

1) if wave propagates so that $\vec{E}$ is along one of the 3 crystal axes, it propagates normally, with a refractive index assoc with that crystal axis

2) for arbitrary directions of $\vec{k}$ and $\vec{E}$ (relative to crystal axes)

$\vec{D} \not\parallel \vec{E}$ and $\vec{k} \not\parallel \vec{S}$

$\vec{H} \perp \vec{D}, \vec{E}, \vec{k}, \vec{S}$

3) the index of refraction varies as an ellipse in between the extreme values of n along the axes.

Vector directions: get these from Maxwell

for a plane wave: all terms $(D, E, B, H) \propto e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$\partial/\partial t \to -i\omega \quad \nabla\cdot \to i\vec{k}\cdot$

$\nabla\cdot D = 0 \to \vec{k}\cdot\vec{D} = 0 \quad \therefore \vec{k} \perp \vec{D}$

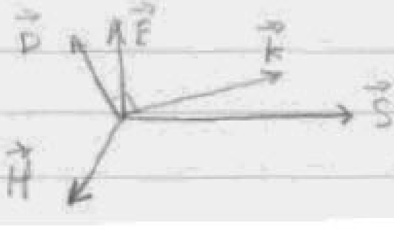
$\nabla\cdot\vec{H} = 0 \to \vec{k}\cdot\vec{H} = 0 \quad \vec{k} \perp \vec{H}$ assume non magnetic

$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \to \vec{k}\times\vec{E} = \omega\vec{B} \quad \therefore \vec{B} \perp \vec{k}, \vec{E}$

$\nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t} \to \vec{k}\times\vec{H} = -\omega\vec{D} \quad \therefore \vec{D} \perp \vec{k}, \vec{H}$

finally $\vec{S} = \vec{E}\times\vec{H}$ Poynting vector gives direction of <u>energy flow</u> = beam direction

wavefronts are $\perp \vec{k}$



Index ellipsoid

- allows calculation of $n(\theta, \phi)$
  - important for tuning phase matching

energy density is a scalar:

$2U_E = \vec{D}\cdot\vec{E}$ this connects the vector components

$D_x = \varepsilon_x E_x$, $D_y = \varepsilon_y E_y$, $D_z = \varepsilon_z E_z$

in crystal coord. system. $\varepsilon_x = \varepsilon_0 n_x^2$, etc

$$\therefore\ 2U_E = \frac{D_x^2}{\varepsilon_x} + \frac{D_y^2}{\varepsilon_y} + \frac{D_z^2}{\varepsilon_z}$$

put into a form like an ellipsoid:

$$\frac{1}{2U_E\varepsilon_0}\left(\frac{D_x^2}{n_x^2} + \frac{D_y^2}{n_y^2} + \frac{D_z^2}{n_z^2}\right) = 1$$

define a unit vector $\hat{d} = \vec{D}/2U_E\varepsilon_0 \rightarrow$ points along $\vec{D}$

$$\frac{\hat{d}_x^2}{n_x^2} + \frac{\hat{d}_y^2}{n_y^2} + \frac{\hat{d}_z^2}{n_z^2} = 1 \quad \text{eqn. of ellipse}$$

For phase matching, we need to know $\vec{k}$ inside crystal

$\rightarrow$ phase, group velocity

let $\hat{k} = \vec{k}/k_0$ normalized to vacuum $k_0 = \omega/c$

suppose $\vec{k}, \vec{D}$ are in x-z plane of crystal $\vec{k}\perp\vec{D}$

> in a uniaxial crystal, $n_z = n_e$ $n_x = n_o$

$\vec{k} = \hat{x}\, k\sin\theta + \hat{z}\, k\cos\theta$

> effective index will be between n_e, n_o

write $k = n_e(\theta) k_0$ find $n_e(\theta)$

$$\vec{D} = -\frac{1}{\omega}\vec{k}\times\vec{H} = -\frac{1}{\omega}\left(\hat{x}\, k_z H_0 + \hat{z}\, k_x H_0\right) \qquad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ k_x & 0 & k_z \\ 0 & H & 0 \end{vmatrix}$$

$$D_x = \frac{k_z}{\omega} H_0 = H_0 \frac{n}{c}\cos\theta \qquad D_z = -\frac{k_x}{\omega} H_0 = -H_0\frac{n}{c}\sin\theta$$

Put these $\vec{B}$ components into ellipsoid eqn.

$$\frac{1}{2U_E\epsilon_0}\frac{H_0^2}{c^2}n^2\left(\frac{\cos^2\theta}{n_o^2}+\frac{\sin^2\theta}{n_e^2}\right)=1$$

now magnetic energy is $U_H = \frac{1}{2}\vec{H}\cdot\vec{B} = \frac{1}{2\mu_0}H_0^2$

So now the multiplier in front is

$$\frac{U_H}{U_E}\frac{1}{\epsilon_0\mu_0c^2} = \frac{U_H}{U_E} = 1$$

equivalence of $U_E = U_H$ is shown on next page.

Finally:

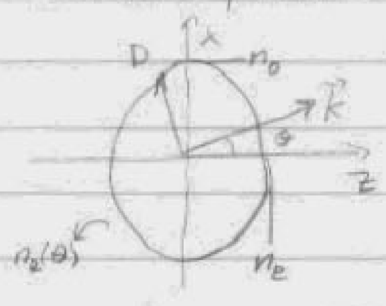
$$\boxed{\frac{1}{n_e^2(\theta)} = \frac{\sin^2\theta}{n_e^2}+\frac{\cos^2\theta}{n_o^2}}$$

Example: uniaxial crystal

1) $\vec{k}$ along $\hat{z}$: both polarizations $\rightarrow n = n_o$
2) tilt crystal θ away from $\hat{z}$ in x-z plane.

D_y still sees n_o

$D_x \rightarrow n_e(\theta)$ given above



where $\vec{D}$ intersects index ellipsoid $\rightarrow n_e(\theta)$

Show that $U_H = U_E$ in an anisotropic medium

(Gaussian units)

energy density in magnetic field:

$$8\pi U_H = \vec{H}\cdot\vec{B}$$

put these in terms of $\vec{E}$ field

$$\vec{k}\times\vec{E} = \frac{\omega}{c}\mu\vec{H} = \frac{\omega}{c}\vec{B}$$

let's write $\vec{k}/(\omega/c) = n\hat{k}$

so $\vec{H}\cdot\vec{B} = \frac{n}{\mu}(\hat{k}\times\vec{E})\cdot n(\hat{k}\times\vec{E})$

vector ID $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

(see Jackson inside cover)

$$\vec{H}\cdot\vec{B} = \frac{n^2}{\mu}\left(E^2 - (\hat{k}\cdot\vec{E})^2\right)$$

Now look at $\vec{E}\cdot\vec{D} = 8\pi U_E$

represent $\vec{D}$ in terms of $\vec{E}$

$$n(\hat{k}\times\vec{E}) = \mu\vec{H}$$

$$n\hat{k}\times(\hat{k}\times\vec{E}) = \mu\hat{k}\times\vec{H} = \frac{1}{n}\mu(-\vec{D})$$

$$n\left(\hat{k}(\hat{k}\cdot\vec{E}) - \vec{E}\right) = -\frac{\mu}{n}\vec{D}$$

$$\therefore\ \vec{E}\cdot\vec{D} = \frac{n^2}{\mu}\left(E^2 - (\hat{k}\cdot\vec{E})^2\right)$$

$\therefore\ \vec{E}\cdot\vec{D} = \vec{H}\cdot\vec{B}$ and $U_E = U_H$

Birefringent phase matching

It is clear from the solutions of the coupled wave eqns. that high efficiency requires

$$\Delta k = k_1 + k_2 - k_3 = 0$$

When $\Delta k \neq 0$ the yield varies as

$$\mathrm{sinc}^2(\Delta k L/2)$$

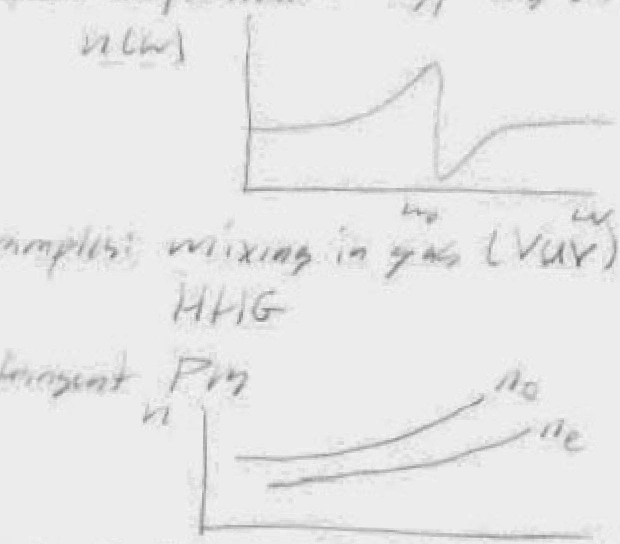
$\Delta k = 0$ sets a condition on the refr. index:

$$\Delta k = k_1 + k_2 - k_3 = n_1 \omega_1 + n_2 \omega_2 - n_3 \omega_3$$

as shown in book (2.7.7) this cannot take place with "normal" dispersion, where $n_3 > n_1, n_2$

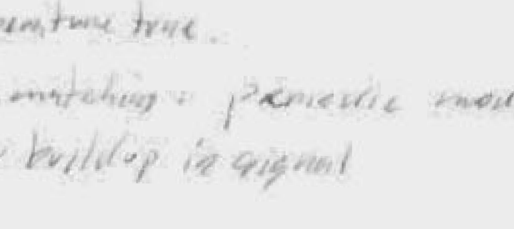
techniques:

- anomalous dispersion: typ ω_3 on high side of resonance.



examples: mixing in gas (VUV)
HHG

- birefringent PM



neg. uniaxial

> angle-tune
> temperature tune.

- Quasi phase matching: periodic modulation at $\Delta k \neq 0$
  → slow buildup in signal

Refraction into birefringent materials

(Guenther appx 13C)

since $E_{\parallel}$ is conserved across boundary

$$(\vec{E}_1)_{\parallel}\, e^{i(\vec{k}_1\cdot\vec{r} - \omega t)} = (\vec{E}_2)_{\parallel}\, e^{i(\vec{k}_2\cdot\vec{r} - \omega t)}$$

where $\vec{r}$ is in x-y plane (on surface)

$\therefore$ as usual $\vec{k}_1 \cdot \vec{r} = \vec{k}_2 \cdot \vec{r}$

$$k_0 n_1 \sin\theta_1 = k_0 n_2 \sin\theta_2$$

for $n_2 = n_o$ this is normal Snell's law

for $n_2 = n_e(\theta_2)$,

$\rightarrow$ more complicated relationship

use $n_e(\theta_2) = \left(\dfrac{\sin^2\theta_2}{n_e^2} + \dfrac{\cos^2\theta_2}{n_o^2} \right)^{-\frac{1}{2}}$

Birefringent case: most common by far.

2 options — ω_2 is along lowest index direction.

Type I:

ω_1 and ω_2 share same polarization.

$\vec{E}_1 \parallel \vec{E}_2$

Type II:

$\vec{E}_1 \perp \vec{E}_2$

"Type III" also called 90° Phase matching

- like type I, but input at $\theta = 90°$ and temperature tuned.
- LiNbO$_3$ most common.

Angle tuning:

use $n_e(\theta)$ function

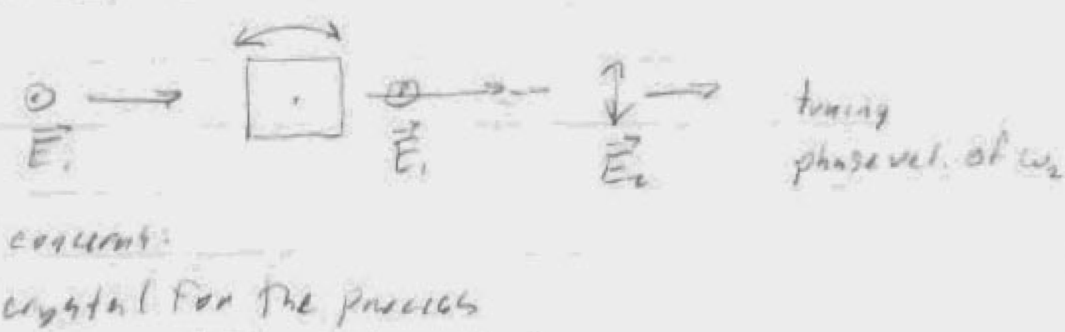
$$\frac{1}{n_e(\theta)^2} = \frac{\sin^2\theta}{n_e^2} + \frac{\cos^2\theta}{n_o^2}$$

__Ex__ type I: neg. uniaxial ($n_e < n_o$) doubling $\Delta k = 2k_1 - k_2 \Rightarrow \frac{2\omega_1}{c}(n_1 - n_2)$

$n_e(2\omega, \theta) = n_o(\omega)$

input $\vec{E}_1$ along n_o, vary crystal angle around $\hat{E}_1$ to tune index of ω_2

$\vec{E}_2 \perp \vec{E}_1$



practical concerns:

- cut crystal for the process
  - x-ray diff to get angle close
- protective A/R coating or in cell to keep off water.

- wavelength separation:
  - prism: no background, – polarizer (too expensive)
  - dichroic mirror: typ. refl harmonic at "S"

    transm fundam. at "P"

for best efficiency

often need multiple mirrors to get rid of fundam.

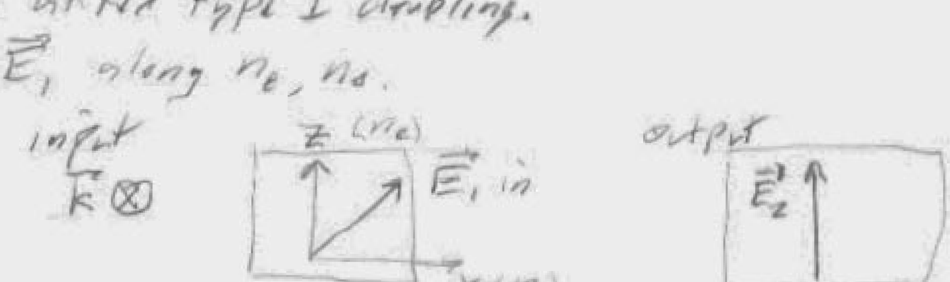
- temperature stabilization.
  - heating to drive off moisture.
  - harmonic is often partly absorbed in crystal

    $\rightarrow \Delta T \rightarrow \Delta k$

    $\therefore$ stabilize at $T > T_{room}$.
  - can fine tune Δk with ΔT

Type II: doubling $\Delta k = \frac{1}{c}\left(\omega_1 n_e(\theta,\omega_1) + \omega_1 n_o(\omega_1) - \omega_2 n_e(\theta,\omega_2)\right)$

- often seen with KTP higher nonlinearity.
- tripling after type I doubling.
- equal $\vec{E}_1$ along n_e, n_o.



$\therefore$ output at 45° to input

mirrors out of plane used to flip polarization.
(typically)