The Language of and Solutions to Linear Systems

Text: 1.3-1.5, 1.7 Section Overviews: 1.3-1.5, 1.7

Quote of Homework Two Solutions	

Don Juan: Today I am neither a warrior nor a *diablero*. For me there is only the traveling on the paths that have a heart, on any path that may have a heart. There I travel, and the only worthwhile challenge for me is to traverse its full length.

Carlos Castaneda - The Teachings of Don Juan: A Yaqui Way of Knowledge (1968)

1. Warm Ups

1.1. **Existence and Uniqueness of Solutions and Pivot Structure.** Determine the values of h and k so that,

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & h & k \end{array}\right],$$

(1) Is consistent (has a solution). [1]
(2) Is consistent with a unique solution. [2]
(3) Is inconsistent.

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & h & k \end{array}\right] \sim^{R3=R3-3R1} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{array}\right]$$

corresponds to the linear system

$$x_1 + 3x_2 = 2$$
$$(h-9)x_2 = k-6$$

For infinitely many solutions (for x_2 to be a free variable) we require that

$$(h-9)x_2 = k-6 \Leftrightarrow 0 \cdot x_2 = 0 \Rightarrow h = 9, k = 6$$

Thus x_2 is free.

For this system to be consistent with a unique solution,

$$(h-9)x_2 = k-6 \Rightarrow x_2 = \frac{k-6}{h-9}, \text{ assuming } h-9 \neq 0$$

thus $h-9 \neq 0 \Rightarrow h \neq 9$ will yield no free variables and the linear system is consistent with a unique point of intersection of the two lines.

For no solutions we require,

$$(h-9)x_2 = k-6 \Leftrightarrow 0 \cdot x_2 = c, c \in \Re, c \neq 0$$

This implies that $h = 9$ and $k \neq 6$. Thus the augmented column is a pivot column and the system has no solutions.

[1] A system is consistent if there are no inconsistent rows.

[2] A system is consistent with a unique solution if there are no inconsistent rows and there are as many pivots as variables.

1.2. **Coefficient Data and Existence and Uniqueness of Solutions.** Assuming that $a \neq 0$, find an equation that restricts a, b, c, d so that the following system has only the trivial solution. [3]

$$ax_1 + bx_2 = 0 \tag{1}$$

$$cx_1 + dx_2 = 0. \tag{2}$$

This system corresponds to the following augmented system,

$$\left[\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} ac & bc & 0 \\ ac & ad & 0 \end{array}\right] \tag{3}$$

$$\sim \left[\begin{array}{cc|c} ac & bc & 0 \\ 0 & ad-bc & 0 \end{array}\right], \tag{4}$$

whose echelon form implies that for there two be a pivot for every variable $ad - bc \neq 0$. This number characterizes the unique solubility of the linear system. Every square matrix has such a number, which is generally called a determinant.

1.3. **Language of Vector Equations.** Given,

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 6 \\ -7 & 5 & 3 \\ 9 & -3 & 3 \end{bmatrix},$$

and observe that the first column plus twice the second column equals the third column. Find a nontrivial solution to the associated homogeneous system. [4]

The homogeneous problem for $\mathbf{A}$ has the two equivalent forms,

$$\mathbf{A}\mathbf{x} = \sum_{i=1}^{3} x_i \mathbf{a}_i = \mathbf{0}. \tag{5}$$

Notice that we are given that, $\mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$. Thus, we are also given one non-trivial solution of $\mathbf{x} = [1\,2\,-1]^{\mathrm{T}}$. To find all non-trivial solutions is a matter of row-reduction.

2. The Vector Equation: $\sum_{j=1}^{n} x_j \mathbf{a}_j = \mathbf{b},\ \mathbf{a}_j, \mathbf{b} \in \mathbb{R}^m$

Given,[5]

$$\mathbf{A}_1 = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix},$$

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix},$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

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[3] Hint: Find the echelon form of the associated matrix equation and from this echelon form find a relation/rule involving a, b, c, d so that the augmented matrix has as many pivots as variables. If you have taken the *determinant* of 2×2 matrices then you can check your work.

[4] Hint: You could solve the system $\mathbf{A}\mathbf{x} = \mathbf{0}$ and find ALL solutions to the homogeneous problem but you aren't asked for this. The idea here is that you could avoid row-reduction altogether. How? Well, write the system in its associated vector form and try to find the vector's coefficients.

[5] All of the following problems require you to apply row-reduction to the appropriate augmented matrix and then interpreting the results. You have already done the row-reductions in homework1, which means that all you have to do now is interpret the pivot structure.

Before we get into these problems we record the following row-reductions:

$$[\mathbf{A}_1\,|\mathbf{b}_1\,] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \tag{6}$$

$$\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} 1 & -5 & 1 \\ -1 & 7 & 1 \\ -3 & 8 & h \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2h+20 & 0 \end{array}\right] \tag{7}$$

$$\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3] = \begin{bmatrix} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & h-10 \end{bmatrix} \tag{8}$$

$$[\mathbf{X}|\mathbf{y}] = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3|\mathbf{y}] = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{9}$$

$$[\mathbf{A}_2|\mathbf{b}_2] \sim \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 0 & 20 & 10 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{10}$$

2.1. **Linear Combinations.** Is $\mathbf{b}_1$ a linear combination of the columns of $\mathbf{A}_1$?

Recall the the following equivalence for $\mathbf{A} \in \mathbb{R}^{m\times n}$,

$$\mathbf{A}\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{a}_i, \tag{11}$$

which says that the matrix product is the same as a linear combination of its columns. So, asking if a vector, $\mathbf{b}$, is a linear combination of columns is also asking if $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution. Either way we study the pivot structure of $[\mathbf{A}|\mathbf{b}]$. Thus, from above we have that $\mathbf{A}_1\mathbf{x} = \mathbf{b}_1$ has no solution and therefore $\mathbf{b}_1$ cannot be written as a linear combination of the columns from $\mathbf{A}$.

2.2. **Linear Dependence.** Determine all values for h such that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a linearly dependent set.

A set of n-many vectors, $\mathbf{v}_i$, forms a linearly independent set if and only if $c_i = 0$ for $i = 1, 2, 3, \ldots, n$ is the only solution to $\sum_{i=1}^{n} c_i \mathbf{v}_i = \mathbf{0}$. This is equivalent to asking if $\mathbf{V}\mathbf{c} = \mathbf{0}$ has only the trivial solution, where $\mathbf{V}$ is a matrix formed by the set of vectors. If a homogeneous system has the trivial solution then there must be a pivot for every variable. If we want the vectors to form a linearly **de**pendent set then we must have the existence of free variables. Thus, from above, we require $h = -10$.

2.3. **Linear Independence.** Determine all values for h such that $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ forms a linearly independent set.

We repeat the argument of 1.2 and now require a pivot for each variable. In this case we have no values of h such that $c_1 = c_2 = c_3 = 0$ is the only solution to $\sum_{i=1}^{3} c_i \mathbf{w}_i = 0$. Thus, the vectors ALWAYS form a linearly dependent set.

2.4. **Spanning Sets.** How many vectors are in $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$? How many vectors are in span(S)? Is $\mathbf{y} \in$span(S)?

This is all a question of language. The set S has three elements. However, the spanning set of S is the set of all linear combinations of the vectors in S. That is, the spanning set of S is every vector that takes the form, $\sum_{i=1}^{3} c_i \mathbf{x}_i$ for any $c_i \in \mathbb{R}$. This spanning set, by definition has infinitely many elements.[6] Finally, we ask if $\mathbf{y}$ is in this spanning set, which is really asking if there are c_i's such that $\mathbf{y}$ can be written as $\sum_{i=1}^{3} c_i \mathbf{x}_i$. Again, this is the same as asking if $\mathbf{X}\mathbf{c} = \mathbf{y}$, which is addressed by understanding the solubility of $[\mathbf{X}|\mathbf{y}]$. From the previous row-reductions we see that this system has a solution, in fact is has many, and therefore $\mathbf{y} \in \text{span}(S)$.

[6] As a simple case consider every scaling of $\hat{\mathbf{i}}$ how many elements would be in this set?

2.5. **Introduction to Matrix Spaces.** Is $\mathbf{b}_2$ a solution to the homogeneous problem of $\mathbf{A}_2$? Is $\mathbf{b}_2$ a linear combination of the columns of $\mathbf{A}_2$?

The null-space of a matrix is the set of all solutions to $\mathbf{Ax} = \mathbf{0}$. This space tells us about all the points in space the homogeneous linear equations simultaneously intersect. One way to determine if $\mathbf{b}_2$ is in the null-space of $\mathbf{A}_2$ is by solving the homogeneous equation and determining if $\mathbf{b}_2$ is one of these solutions. However, it pays to note that if $\mathbf{b}_2$ is in the null-space of $\mathbf{A}_2$ then $\mathbf{A}_2\mathbf{b}_2 = \mathbf{0}$. A quick check shows,

$$\mathbf{A}_2\mathbf{b}_2 = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \mathbf{0}. \tag{12}$$

The column space, on the other hand, is a little different. The column space is the set of all linear combinations of the columns of $\mathbf{A}_2$. This is also called the spanning set of the columns of $\mathbf{A}_2$. Thus, this question can be addressed in the same way as problem 1.1 or the last part of 1.4 and we have,

$$[\mathbf{A}_2|\mathbf{b}_2] \sim \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 0 & 20 & 10 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{13}$$

The conclusion is that $\mathbf{b}_2$ is in both the null-space and column space of $\mathbf{A}_2$. This is not generally true of a non-trivial vector. In fact, it is never true for rectangular coefficient data.

3. The Matrix Equation: $\mathbf{Ax} = \mathbf{b}$, $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$

Given,[5]

$$\mathbf{A}_1 = \begin{bmatrix} 1 & -3 & 0 \\ -1 & 1 & 5 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 6 & 18 & -4 \\ -1 & -3 & 8 \\ 5 & 15 & -9 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 3 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}, \quad \mathbf{A}_5 = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix},$$

$$\mathbf{b}_1 = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 20 \\ 4 \\ 11 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_4 = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}, \quad \mathbf{b}_5 = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix}.$$

3.1. **Algebra.** Find all solutions to $\mathbf{A}_i\mathbf{x} = \mathbf{b}_i$ for $i = 1, 2, 3, 4, 5$.

3.2. **Geometry.** Describe or plot the geometry formed by the linear systems and their solution sets.

The following row-equivalences can be checked via *computational tools*.[7]

$$[\mathbf{A}_1\,|\mathbf{b}_1\,] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \tag{14}$$

$$[\mathbf{A}_2\,|\mathbf{b}_2\,] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{15}$$

$$[\mathbf{A}_3\,|\mathbf{b}_3\,] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \tag{16}$$

$$[\mathbf{A}_4\,|\mathbf{b}_4\,] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{17}$$

$$[\mathbf{A}_5\,|\mathbf{b}_5\,] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \tag{18}$$

Since the reduced row echelon matrices share the same solutions as the original linear systems we have,

$$(14) \implies \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix},$$

$$(15) \implies \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4-3x_2 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4-3t \\ t \\ 1 \end{bmatrix}, \; t \in \mathbb{R},$$

$$(17) \implies \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10-2x_2-3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10-2t-3s \\ t \\ s \end{bmatrix}, \; t, s \in \mathbb{R}.$$

Since in systems three and five the reduced row echelon forms have an inconsistent row their associated systems do not have solutions. More precisely, the first two equations of both systems have points of common intersection but the third equation does not share these points.

3.3. **Geometry.** Describe or plot the geometry formed by the linear systems and their solution sets.

- System one is the algebraic representation of three planes in space, which share a common point of intersection, $(2, -1, 1)$.
- System two is the algebraic representation of three planes in space, which share common points of intersection. There are infinitely many of these points defined by $\mathbf{x} = [4 - 3t\ t\ 1]^{\mathrm{T}}$, which parameterizes a line in space.
- System three is the algebraic representation of three planes in space, which share no common points of intersection. This does not mean that the planes do not intersect one another. It just means that they do not do so simultaneously.
- System four is the algebraic representation of three planes in space, which share common points of intersection. There are infinitely many of these points defined by $\mathbf{x} = [10 - 2t - 3s\ t\ s]^{\mathrm{T}}$, which parameterizes a plane in space.

[7]These calculations should be done by hand. There is no replacement for this type of practice. Computational tools should be used to check your work either as you go or at the end of your hand-calculations. A good tool can be found through the external material links on the ticc website. The tool-kit will row-reduce a matrix and show you the steps it used.

- System five is the algebraic representation of three lines in space, which share no common points of intersection. Again, these lines intersect one another but do not have any points in space where they do so simultaneously.

4. Linear Independence

Given,

(19)
$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 9 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

4.1. **Row-Reduction.** Find the row echelon form of the matrix $\mathbf{V}$ whose columns are the given vectors.

The reduced row-echelon form of the matrix $\mathbf{V} = [\mathbf{v}_1\, \mathbf{v}_2\, \mathbf{v}_3\, \mathbf{v}_4]$ is given by,

(20)
$$\mathbf{V} = \begin{bmatrix} 0 & 3 & -4 & -1 \\ 9 & -4 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 11/38 \\ 0 & 1 & 0 & 5/19 \\ 0 & 0 & 1 & 17/38 \end{bmatrix}.$$

4.2. **Linear Independent Sets.** Does $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ form a linearly independent set? What about the sets $S_2 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, $S_3 = \{\mathbf{v}_1, \mathbf{v}_2\}$, $S_4 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$, $S_5 = \{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Once the row-reduction has been completed it is now a matter of interpretation of the pivot-structure. The key point to note is that we care about the placement of the pivots relative to the columns and not the rows. This is because pivots in every row give us guaranteed existence of solutions to linear systems. Since dependence arguments are given by homogeneous systems, existence of solutions is never a questions. What we are really after is uniqueness of solutions, which has to do with free variables, which has to do with columns. With that said where is what we know,

- Since there is not a pivot in every column of the matrix representation of the set of vectors, S_1 is not a linearly independent set.
- S_2 is a linearly independent set.
- S_3 is a linearly independent set.
- S_4 is a linearly independent set.
- S_5 is not a linearly independent set.

4.3. **Spanning Sets.** Does S_1 span $\mathbb{R}^3$? What about S_2, S_3, S_4, S_5? If you were going to span $\mathbb{R}^3$ then which of these sets would you choose? [8]

The span is the set of all linear combinations of a given set of vectors. In the first case we have,

(21)
$$\text{span}(S_1) = \left\{ \mathbf{b} \in \mathbb{R}^3 : \mathbf{b} = \sum_{i=1}^{3} x_i \mathbf{v}_i, x_1, x_2, x_3 \in \mathbb{R} \right\},$$

and now we consider pivots in rows. Since the matrix representation of S_1 has a pivot in every row, there exist x_1, x_2, x_3 such that the previous linear combination is true for every $\mathbf{b} \in \mathbb{R}^3$. Thus, $\text{span}(S_1) = \mathbb{R}^3$. For the other sets we have,

- $\text{span}(S_2) = \mathbb{R}^3$
- $\text{span}(S_3)$ is a two-dimensional subset of $\mathbb{R}^3$.
- $\text{span}(S_4) = \mathbb{R}^3$
- $\text{span}(S_5)$ is a two-dimensional subset of $\mathbb{R}^3$.

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[8]Remark: This problem is meant to demonstrate a few reoccurring points in linear algebra. The idea is that we typically work problems backwards in the sense that you start with a space of vectors, say $\mathbb{R}^n$, and ask the question,

- Given a 'vector-space' can we 'reach' every 'point' in the space.

More importantly, how can we do this with a minimal set of vectors? In this problem the vector-space is $\mathbb{R}^3$ and from calculus we know that we need only three linearly-independent vectors, typically $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$, to reach every point via linear combination (aka span the space), $\mathbf{x} = x_1\hat{\mathbf{i}} + x_2\hat{\mathbf{j}} + x_3\hat{\mathbf{k}} \in \mathbb{R}^3$. Consequently, if we choose any four vectors they must be linearly-**de**pendent, S_1, which means that some vectors point in redundant directions. However, that doesn't mean that we can pick any three vectors and still span the space, S_5. So, we have to make careful choice to take enough vectors to span the space but **not so many** that the set of vectors is linearly-**de**pendent. When we have made this careful choice we have secretly constructed a coordinate system or basis for the space. This choice is not unique, S_2, S_4.

5. Application

Suppose you have a set S of three points in $\mathbb{R}^2$,

$$S = \{(t_1, p_1), (t_2, p_2), (t_3, p_3)\}, \tag{22}$$

which you seek to interpolate with the quadratic polynomial $p(t) = a_0 + a_1 t + a_2 t^2$.

5.1. Interpolations and Linear Systems.

Using S and $p(t)$ define a linear system of equations in the a_0, a_1, a_2 variables.[9]

First notice that even though the polynomial equation is nonlinear in the t-variable it is linear in the coefficients and $p(t) = \begin{bmatrix}1 \; t \; t^2\end{bmatrix}^{\mathrm{T}} \cdot [a_0 \; a_1 \; a_2]^{\mathrm{T}}$. Every point, $(t_i, p_i = p(t_i))$, given defines a new polynomial and in the case of three points we have the following linear system,

$$\begin{bmatrix} 1 & t_0 & t_0^2 \\ 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix}.$$

5.2. Existence and Uniqueness in Interpolation.

Determine which of the following sets of points can be uniquely interpolated by $p(t)$. [10]

$$S_1 = \{(1, 12), (2, 15), (3, 16)\}$$
$$S_2 = \{(1, 12), (1, 15), (3, 16)\}$$
$$S_3 = \{(1, 12), (2, 15), (2, 15)\}$$

Of the three sets of points only the set S_1 admits a unique interpolating polynomial. This can be seen by putting these values in the matrix representation of the problem. Doing so for S_2, S_3 gives a two equivalent rows in the coefficient matrix, which always leads to a row of zeros.

We can apply some common sense to the geometry of the problem. Looking at S_2 we see that the graph must pass through $(1, 12)$ and $(1, 15)$. The vertical line test tells us no function can do this. Looking at S_3 we see that the second and third point are the same. A quick reduction shows that there is a free-variable in the augmented matrix. This implies that while there is only one line that connects two points in space there are many quadratic polynomials that connect two points in space.

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[9]Hint: This problem is meant to trick you. Clearly, $p(t)$ is a nonlinear equation in the t-variable but once you have chosen a t-value then it is a linear equation in the coefficient variables. If you choose many t-values then you have many linear equations and now the tools of linear algebra apply.

[10]Your choice! We have two ways to approach this problem. First, you have a linear system and thus row-reduction and interpretation of pivot structure. However, if you think about the geometry of the points and the possible graphs of quadratic polynomials you should be able to determine, which of the points can be interpolated.