The reflection coefficients for the parallel polarization components from the two interfaces are set equal to each other

$$\frac{\tan(\theta_i - \theta_{t1})}{\tan(\theta_i + \theta_{t1})} = \frac{\tan(\theta_{t1} - \theta_{t2})}{\tan(\theta_{t1} + \theta_{t2})}$$

This equality insures complete cancellation of the parallel component of polarization. After applying these design rules, the reflectivity of the normal component of polarization from the dielectric layer is

$$R = \frac{(n_1^2 - 1)\sin^2\theta_i - \cos\theta_i\sqrt{n_2^2 - \sin^2\theta_i}}{(n_1^2 - 1)\sin^2\theta_i + \cos\theta_i\sqrt{n_2^2 - \sin^2\theta_i}}$$

Interference polarizers are used in laser systems when the incident radiation will strike the dielectric layer at an angle. The polarizing beam splitter is a popular example of this design. The radiation is incident on the dielectric layer at an angle of 45°. The wave transmitted by the layer is polarized with its electric vector in the plane of incidence, whereas the reflected wave has its electric vector normal to the plane of incidence.

POLARIZATION BY BIREFRINGENCE

In crystals with cubic symmetry, the propagation properties of light are isotropic, there is a single index of refraction, and the crystal behaves optically like a noncrystalline material such as glass (at least to first order). All other classes of crystals are optically anisotropic. The index of refraction (and naturally, the dielectric constant) depends on the direction of propagation of the light, relative to the crystal axes. We must use tensor calculus (see Appendix 13-A) to discuss the propagation of light in these materials. In general, to analyze the propagation of light in an anisotropic medium, the wave is divided into two waves with orthogonal polarizations (the polarization direction in the medium will be denoted by the direction of the displacement **D**). The two waves, with orthogonal polarizations $\mathbf{D}_1$ and $\mathbf{D}_2$, will not have the same propagation vector and will exhibit double refraction. There are, however, special directions in an anisotropic crystal for which both polarizations have the same propagation velocity. The special direction is called an *optical axis*, and if two such directions exist, the crystals are called *biaxial*. If there is only one direction for which the two orthogonal polarizations have the same propagation velocity, the crystal is called *uniaxial*.

The physical origins of birefringence can be understood by modifying the classical model of dispersion introduced in Chapter 7. The single spring constant used in the model **(7-24)**, can be replaced by three spring constants, one for each of the coordinate directions. The application of the procedure used in Chapter 7 to this modified model leads to three indices of refraction for the material, that is, a biaxial crystal; examples are listed in Table 13.2. If the spring constants are equal in two coordinate directions, then the model predicts two indices of refraction, that is a uniaxial crystal, (see Table 13.3). If a different damping term for each coordinate direction is included in the model **(7-26)**, the result is a complex index of refraction that has directional dependence. This model would describe pleochroism.

To help understand the physical origin of the different coupling constants, consider the unit cell of the uniaxial crystal calcite shown in Figure 13-7. A crystal of calcite is made up of a three-dimensional array of these

TABLE 13.2 Biaxial Crystals

Mineral	Index of Refraction n_α	n_β	n_γ
Tridymite	1.469	1.47	1.473
Mica(muscovite)	1.5601	1.5936	1.5977
Turquoise	1.61	1.62	1.65
Topaz	1.619	1.62	1.627
Sulfur	1.95	2.043	2.240
Borax	1.447	1.47	1.472
Lanthanite	1.52	1.587	1.613
Stibnite(Sb_2S_3)	3.194	4.303	4.46

TABLE 13.3 Uniaxial Crystals

Mineral	Index of Refaction (Na-D) n_o	n_e
Ice(H_2O)	1.309	1.313
Sellaite(MgF_2)	1.378	1.390
Quartz	1.54424	1.55335
Wurtzite(ZnS)	2.356	2.378
Rutile(TiO_2)	2.616	2.903
Cinnabar(HgS)	2.854	3.201
Calcite($CaO{\cdot}CO_2$)	1.658	1.486
Tourmaline	1.669	1.638
Sapphire	1.7681	1.7599

unit cells. Each carbon atom in the crystal lies at the center of an imaginary equilateral triangle with oxygen atoms at each corner. The direction, normal to the planes containing these triangles, is the optical axis of calcite. From the arrangement of atoms, it seems reasonable to assume that the binding energy in the planes containing the oxygen atoms is different than the binding energy normal to these planes. When light is incident on a crystal of calcite parallel to the optical axis, shown in Figure 13-7, the electric displacement of the light wave lies in the same plane as the oxygen atoms. The electric polarization induced by the electric field is the same for all polarization directions in this case and the resulting index of diffraction would be independent of polarization direction.

When a light wave is incident on the calcite crystal normal to the optical axis, the electric polarization induced by the light wave's electric field in the plane of the oxygen atoms is different than the polarization induced by the electric field normal to that plane, resulting in two indices of refraction.

A light wave propagating in an arbitrary direction through this crystal can be decomposed into two waves, with orthogonal polarizations, propagating at different velocities. Any wave whose electric displacement lies in the plane of the oxygen atoms, and thus is perpendicular to the optical axis, obeys Snell's law. For this polarization orientation, the propagation velocity is isotropic and the wave is called the *ordinary wave*. Any other wave with its electric displacement at an angle to the plane containing the oxygen atoms has an anisotropic propagation velocity and does not obey Snell's law. This wave is called the *extraordinary wave* (see Figure 13-8).

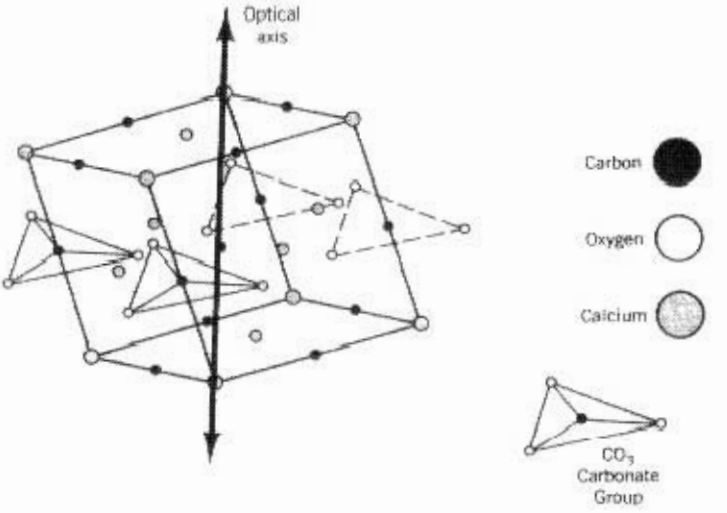


FIGURE 13-7. The unit cell of calcite. Only four of the carbonate groups that make up the unit cell are shown in their entirety; the other groups are represented by the carbon atoms at their center.

The electrons can move quite easily in the planes containing the oxygen atoms so the spring constant for these atoms is small relative to the direction normal to the planes. In the direction normal to the oxygen planes, the electrons experience strong binding forces. For this reason, the propagation velocity in the direction of the optical axis, the ordinary wave's velocity, is less than the propagation velocity normal to that direction. The index of refraction associated with the ordinary wave is thus greater than the index associated with the extraordinary wave, $n_0 > n_e$. Crystals with this property are called *negative* uniaxial crystals ($n_e - n_0 < 0$). The polarization direction for the ordinary wave of a negative, uniaxial crystal is called the *slow axis*. The polarization direction (the direction of **D**) for the wave that has the

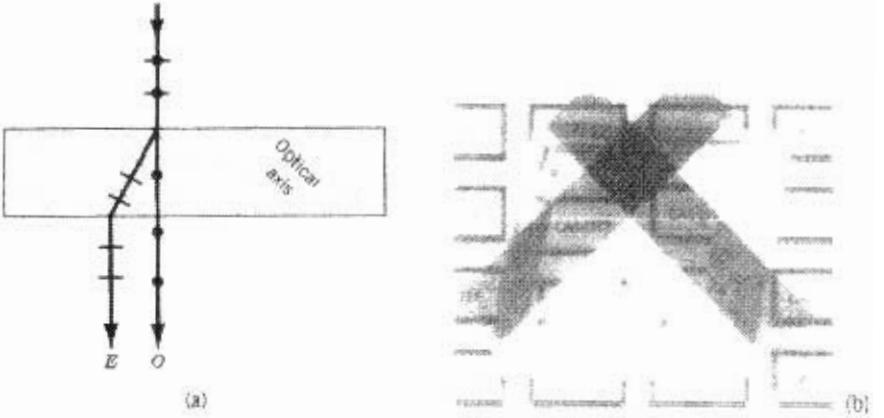


FIGURE 13-8. (a) Uniaxial crystal showing the failure of the extraordinary wave to obey Snell's law. The lines in the crystal indicate the direction called the optical axis. (b) The birefringence shown in (a) results in a double image. The polarizations of the two images are orthogonal, as indicated by the images transmitted by the two Polaroid strips. The long axis of each strip is parallel to the polarization axis.

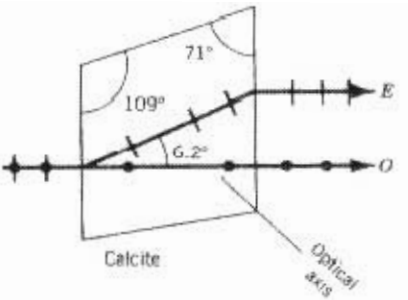


FIGURE 13-9. Separation of the extraordinary (the upper beam) and ordinary (lower beam) waves in calcite. The line labeled optical axis indicates the direction of the optical axis. The rhombohedron shown is the naturally occurring form of a single crystal of calcite.

higher propagation velocity is called the *fast axis*, and for a *positive* uniaxial crystal ($n_e - n_0 > 0$), this wave is the ordinary wave.

In general, when an unpolarized beam of light is incident normal to the surface of a plane parallel plate of an uniaxial crystal, there will be two beams emerging from the back side of the crystal, as is shown in Figure 13-8. The ordinary wave, labeled *O* in Figure 13-8, is polarized with its displacement vector **D** normal to the plane containing the optical axis. The extraordinary wave labeled *E* in Figure 13-8 is polarized with its displacement vector in the plane containing the optical axis. The two waves indicated in Figure 13-8*a* produce two images with perpendicular polarizations, as shown in Figure 13-8*b*. Polarizers are designed to utilize the spatial separation of the two beams shown in Figure 13-8 to select a desired polarization direction.

Polarizers are constructed using the uniaxial material; the simplest design utilizes the naturally occurring birefringent crystal calcite and a set of "stops" to remove either of the two beams. See Figure 13-9. Because the separation of the two waves is small (only 6.2° in calcite), this technique can only be used with very narrow beams.

The Wollaston and Rochon polarizers increase the separation of the two beams over that obtainable with a single crystal by using two single crystals, usually quartz, cut and polished into two prisms (see Figure 13-10). The Rochon polarizer has an entrance prism with its optical axis oriented perpendicular to the incident face of the polarizer in the plane of Figure 13-10 and a second exit prism, glued to the first with its optical axis perpendicular to the first prism's axis, perpendicular to the plane of Figure 13-10.

The light wave whose polarization is perpendicular to the second prism's axis, and in the plane of Figure 13-10, sees no index discontinuity at the interface between the two prisms because the wave also has its polarization perpendicular to the optical axis of the first prism. For this reason, the ordinary wave is not deviated by the Rochon polarizer. The extraordinary wave of the second prism with its polarization parallel to the optical axis, however, is an ordinary wave in the first prism. It sees a discontinuous change in the index of $(n_e - n_0)$ as it crossed the boundary between the two prisms. The prism angle θ and the difference between the extraordinary and ordinary indices determine the angle φ between the *E* and *O* beams. The Wollaston polarizer produces twice the deviation as the Rochon polarizer by causing both the ordinary and extraordinary wave to see the same discontinuous change in the index. This occurs because the extraordinary wave and ordinary waves interchange roles in the two prisms. The advantage of the Rochon prism is that it keeps the ordinary wave on the optical axis

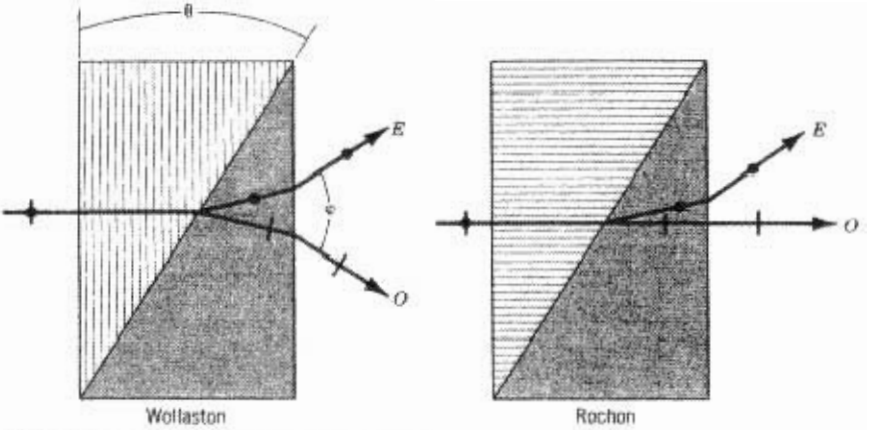


FIGURE 13-10. Two polarizers designed to increase the separation of the extraordinary and ordinary waves. The entrance prism on both polarizers has its ordinary wave polarized out of the paper. The second prism has its ordinary wave polarized in the plane of the paper. The lines and dots used to shade the two prisms indicate the direction of the optical axis.

Another method of separating the two orthogonal polarizations in a birefringent crystal is the use of total reflection to deflect one beam toward an absorber; see Figure 13-11. The Nicol prism named after **William Nicol (1768–1857)** was the first of this type of polarizer. A calcite crystal is cut into two prisms and polished to obtain the angles shown in Figure 13-11. The two prisms are then glued together with a cement called Canada balsam. The cement used to assemble the two prisms is selected so that the ordinary wave experiences total reflection at the glue joint, but the extraordinary wave passes through the polarizer. (Just as is the case with an optical fiber, which depends on total reflection, the Nicol polarizer will operate only with rays arriving within a cone whose angle, called the acceptance angle, is 24°. The acceptance angle of this type of polarizer is determined by the difference between the critical angle of the ordinary and extraordinary rays.) The cement restricts the wavelength range of the polarizer so designs were developed that eliminated the cement.

The Glan–Foucault is a more modern polarizer design (see Figure 13-12) based on the same principal as the Nicol. Its advantages are that it replaces glue with air at the interface of total reflection and the incident and exit faces are perpendicular to the light wave. The acceptance angle of this polarizer is only 7°, but it will operate from 0.23 to 5.0 μm. If the

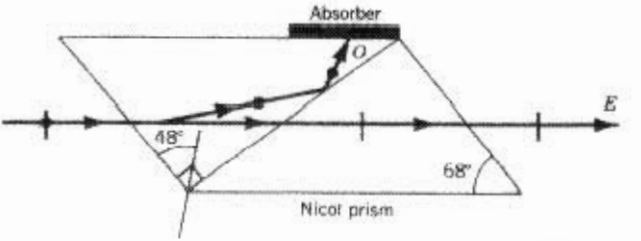


FIGURE 13-11. The geometry of a Nicol prism polarizer. The direction of the optical axis is indicated by the line in the lower left corner.

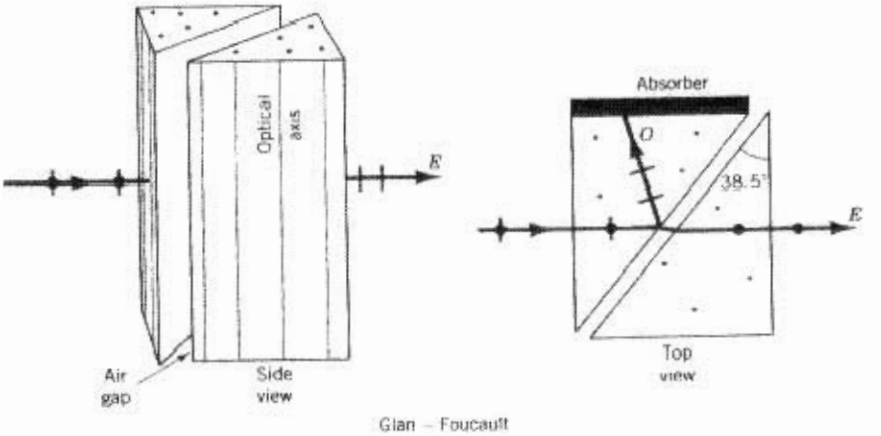


FIGURE 13-12. Two views of the Glan-Foucault polarizer showing its operation. The parallel lines indicate the direction of the optical axis.

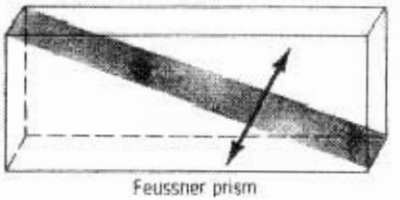


FIGURE 13-13. A Feussner polarizer. The optical axis of the birefringent material is normal to the slab of material as shown by the arrows.

two prisms of the Glan–Foucault polarizer are cemented together, the device is called a Glan–Thompson polarizer. This design modification increases the field of view to 30°, but it cannot be used in the uv.

One of the problems in constructing polarizers using birefringent material is finding or growing a birefringent crystal of a size and optical quality to be useful. One polarizer design uses isotropic prisms with a thin plate of birefringent material sandwiched between the prisms; see Figure 13-13. The two prisms are made of a glass with an index equal to the higher index of the birefringent material. If the birefringent material is calcite, then the ordinary ray is transmitted. The optical axis can be oriented normal to the slab (Feussner polarizer) or parallel to the entrance face (Bertrand type). (Polarizers in optics are like beakers in chemistry; even a slight modification of a design results in a new device bearing the designer's name.)

OPTICAL INDICATRIX

In the previous discussions about light waves, we assumed that the medium through which the light propagated was homogeneous and isotropic. In those cases, we could relate the electromagnetic field vectors **E** and **B** to the electric displacement **D** and the magnetic vector **H** through the scalar quantities ϵ (dielectric constant) and μ (magnetic permeability). We will continue to assume that the material is isotropic with regard to the magnetic field, so that μ remains a scalar, but we will now assume that the dielectric properties can vary with direction. The impact of discarding the assumption of isotropic electrical properties is the need to use tensor relationships between **E** and **D** because the electric field **E** will, in general, not be parallel to **D**.

The complexity of the tensor can be reduced by assuming that there

are no losses in the medium. We will show that this assumption also allows the use of a geometrical construction to aid in understanding the properties of the anisotropic medium. The geometrical construction, called the optical indicatrix, demonstrates the origin of two propagation velocities in the anisotropic material and can be used to determine **E**, **k**, and **S** from the specification of **D**.

The simple relationship between **D** and **E** that we have used up to now must be replaced by the more general equations

$$\begin{aligned} D_x &= \epsilon_{xx} E_x + \epsilon_{xy} E_y + \epsilon_{xz} E_z \\ D_y &= \epsilon_{yx} E_x + \epsilon_{yy} E_y + \epsilon_{yz} E_z \\ D_z &= \epsilon_{zx} E_x + \epsilon_{zy} E_y + \epsilon_{zz} E_z \end{aligned} \tag{13-1}$$

or in more compact notation,

$$D_i = \sum_{j=1}^{3} \epsilon_{ij} E_j$$

The ϵ_{ij} are the components of a tensor of second rank. In general, there would be nine such components, but energy considerations reduce the number of independent elements in the dielectric tensor to six or less. To prove this statement, we must make several assumptions

1. The energy of the electromagnetic field that we used in Chapter 2 **(2-21)** for an isotropic medium is valid for an anisotropic medium.
2. The energy flux (Poynting vector) that we used in Chapter 2 **(2-22)** for an isotropic medium is valid for an anisotropic medium.
3. There is no energy loss in the medium.

The Poynting vector **(2-22)** is used to obtain the unit energy flow into a unit volume

$$\nabla \cdot \mathbf{S} = \nabla \cdot (\mathbf{E} \times \mathbf{H}) \tag{13-2}$$

We use the identity **(2A-13)** to rewrite **(13-2)** as

$$\nabla \cdot \mathbf{S} = \mathbf{E} \cdot \left(\frac{\partial \mathbf{D}}{\partial t} \right) + \mathbf{H} \cdot \left(\frac{\partial \mathbf{H}}{\partial t} \right) \tag{13-3}$$

$$\nabla \cdot \mathbf{S} = \left(\sum_{k=1}^{3} \sum_{j=1}^{3} E_k \epsilon_{kj} \frac{\partial E_j}{\partial t} \right) + \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t}$$

where we assume that the components of the dielectric tensor are independent of time.

We are only interested in the electric energy flux that is the first term of **(13-3)**. The steady-state value of the electric energy flux is obtained by applying **(2-26)** to write

$$\langle \nabla \cdot \mathbf{S} \rangle_E = \frac{1}{2} \left(\sum_{j=1}^{3} E_j \sum_{k=1}^{3} \epsilon_{jk}^{*} \frac{\partial E_k^{*}}{\partial t} + \sum_{i=1}^{3} E_i^{*} \sum_{k=1}^{3} \epsilon_{ik} \frac{\partial E_k}{\partial t} \right) \tag{13-4}$$

Using **(2-14)**, we can rewrite **(13-4)** as

$$\langle \nabla \cdot \mathbf{S} \rangle_E = \frac{i\omega}{2} \sum_{j=1}^{3} E_j \sum_{k=1}^{3} E_k^{*} (\epsilon_{kj} - \epsilon_{jk}^{*}) \tag{13-5}$$

In many books, the Einstein notation is used to represent tensor equations such as **(13-1)**. In this notation,

$$D_i = \epsilon_{ij} E_j$$

The summation sign is supressed and the summation is indicated by the repeating index j. We will not use this notation.

The study of anisotropy is confusing because different names are used to indicate the optical indicatrix. Some of these are listed in Table 13.4.

TABLE 13.4 Names for Optical Indicatrix

Index ellipsoid
Optical indicatrix
Reciprocal ellipsoid
Poinsot ellipsoid
Ellipsoid of wave normals

A second confusing fact is that the optical indicatrix is not the only surface used in the description of optical anisotropy. The first step in reading the optical anisotropy literature is to determine what surface the author is discussing. Other surfaces used in the discussion of optical anisotropy are introduced in the appendices.

We now use the third assumption that the medium is a lossless one, which means that **(13-5)** will be equal to zero, i.e., there is no change in energy flow through the medium. This results in the requirement that

$$\epsilon_{jk}^{*} = \epsilon_{kj}$$

Thus the dielectric tensor $\bar{\epsilon}$ must be Hermitian.

We will treat the dielectric tensor as real, resulting in the requirement that the tensor $\bar{\epsilon}$ must be symmetric. The assumption of a lossless medium reduces the maximum number of tensor components that must be considered to six. A further reduction in the number of components to three can be made by the proper choice of the coordinate system. In the properly oriented coordinate system (with coordinate axes parallel to the *principal dielectric axes*), the nonzero tensor components are called the *principal dielectric constants*. As discussed in Appendix 13-A, the assumption of a lossless medium allows the use of a geometrical surface to aid in understanding the properties of the tensor. The geometrical surface used to represent a second-order, symmetric, dielectric tensor is called the *optical indicatrix*.

Starting with the equation for the electric energy density

$$2U_E = \mathbf{D}\cdot\mathbf{E}$$

(2-21), we select as the coordinate system the principal dielectric axes, allowing us to write

$$D_x = \epsilon_x E_x, \qquad D_y = \epsilon_y E_y, \qquad D_z = \epsilon_z E_z$$

In this coordinate system, we may rewrite the energy density equation as

$$\frac{D_x^2}{2U_E\epsilon_x} + \frac{D_y^2}{2U_E\epsilon_y} + \frac{D_z^2}{2U_E\epsilon_z} = 1 \tag{13-6}$$

Equation **(13-6)** is the equation of an ellipsoid **(13A-5)**, whose semiaxes are equal to the square roots of the principal dielectric constants, or equivalently, the principal indices of refraction. This ellipsoid (see Figure 13A-1) is called the optical indicatrix.

The ellipsoid defined by **(13-6)** is not a representative of the dielectric tensor but its reciprocal called the *impermeability tensor*, defined by the relation

$$E_i = \sum_{j=1}^{3} \eta_{ij} D_j$$

The dielectric tensor is discussed in Appendix 13-B; see **(13B-4)**.

The optical indicatrix can be used to determine **E**, **k**, and **S**, given **D**. The geometrical construction shown in Figure 13-14 will demonstrate the method used to determine these parameters. Assume that **D** is known and represented by the vector from O to P, where P is a point on the ellipsoid in Figure 13-14. Construct first a surface, tangent to the ellipsoid at P, with a normal $\hat{\mathbf{n}}$; then construct the line OQ to be parallel to the surface normal $\hat{\mathbf{n}}$ at P. The point Q is in the plane tangent to P; see Figure 13-14. The magnitude of **E** is given by $1/OQ$ where OQ is the distance from the origin to the plane tangent to P. The direction of **E** is parallel to $\hat{\mathbf{n}}$ and the direction of the Poynting vector **S** is parallel to the line PQ in Figure 13-14.

If the propagation vector **k** of a wave is known, we can determine **D** and **E**. This is accomplished by constructing a plane through the origin normal to

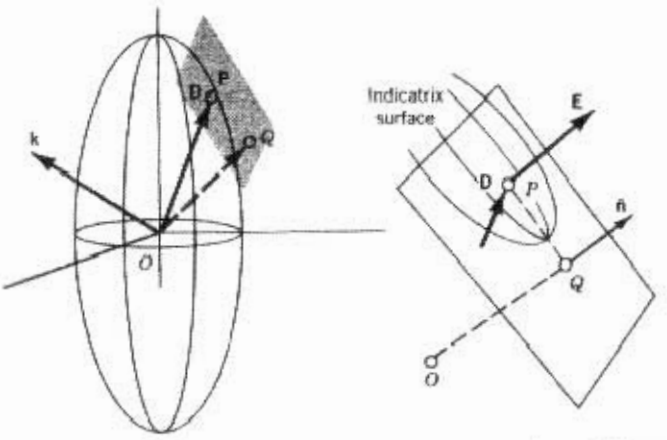


FIGURE 13-14. Use of the optical indicatrix to find the electric field given D.

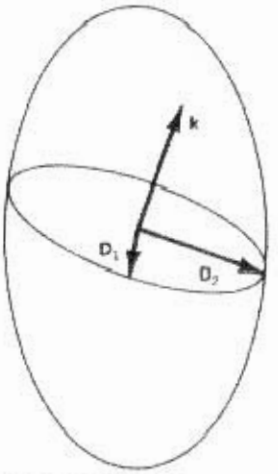


FIGURE 13-15. The determination of D given the propagation vector k.

k (see Figure 13-15). The curve formed by the intersection of the ellipsoid and plane perpendicular to **k** is an ellipse. The principal semiaxes of the ellipse are proportional to indices of refraction n_1 and n_2 (or equivalently, to the reciprocals of the phase velocities)

$$n_1 = \frac{c}{v_{p1}}, \qquad n_2 = \frac{c}{v_{p2}}$$

The directions of the principal semiaxes coincide with $\mathbf{D}_1$ and $\mathbf{D}_2$ that are the two orthogonal polarizations for the wave, with wave vector **k**.

For certain directions of **k**, the plane normal to **k** will cut the ellipsoid so as to form an intersecting curve that is a circle. These special directions are called the *optic axes* (of the wave normals) of the crystal. If there is only one such direction, the crystal is uniaxial. The maximum number of directions that can be found is two. Crystals with two optical axes are biaxial.

FRESNEL'S EQUATION

In this section, we will examine the propagation of a plane wave in an electrically anisotropic medium; we continue to assume that all of the media under consideration are magnetically isotopic so that $\mathbf{B} = \mu_0\mathbf{H}$. The analysis will parallel the one made in Chapter 2 of an isotropic medium. We will discover that **D** remains perpendicular to **k**, but that **E** no longer is perpendicular to **k**. The nonzero, scalar product between **E** and **k** produces an equation, called Fresnel's equation, that predicts the medium will have two indices of refraction associated with two electric displacement vectors. We conclude the section by proving that the two displacement vectors are mutually orthogonal.

We will continue to use the assumption that the medium is nonconducting and that there are no currents or charges present. If we allowed a conductive medium, the analysis would also apply to dichroic materials. Maxwell's equations, as given by **(2-6)**, no longer apply because the constitutive relation **(2-6f)**, $\mathbf{D} = \epsilon\mathbf{E}$, no longer holds. The modified equations are

$$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\cdot\mathbf{D} = 0$$

$$\nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t}, \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{13-7}$$

We will interpret Maxwell's equations, as we did in Chapter 2, by the use of plane waves

$$\mathbf{E} = \mathbf{E}_0 \exp[i(\omega t - \mathbf{k}\cdot\mathbf{r} + \phi)], \qquad \mathbf{D} = \mathbf{D}_0 \exp[i(\omega t - \mathbf{k}\cdot\mathbf{r} + \phi)]$$

$$\mathbf{B} = \mathbf{B}_0 \exp[i(\omega t - \mathbf{k}\cdot\mathbf{r} + \phi)], \qquad \mathbf{H} = \mu\mathbf{B}$$

Transverse Waves

Substituting the plane wave solutions into the first pair of Maxwell's equations yields, for the electric field

$$\nabla\cdot\mathbf{D} = -i\mathbf{k}\cdot\mathbf{D} = 0$$

D remains perpendicular to k, but this fact no longer implies that E is perpendicular to k. For the magnetic field,

$$\nabla\cdot\mathbf{B} = \mu\nabla\cdot\mathbf{H} = -i\,\mathbf{k}\cdot\mathbf{H} = 0$$

Since we continue to assume that the material is magnetically isotropic, we still find that both B and H are perpendicular to k.

Interdependence of D and H

Continuing to examine Maxwell's equations, we substitute the plane wave equations into the second pair of equations

$$\nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t}, \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$i\,\mathbf{k}\times\mathbf{H} = -i\omega\,\mathbf{D}, \qquad i\,\mathbf{k}\times\mathbf{E} = i\omega\,\mathbf{B} = i\omega\mu_0\mathbf{H}$$

The plane wave solutions of Maxwell's equations require that H be perpendicular to both k and D, and also H must be perpendicular to both k and E.

Fresnel's Equation

The fact that E may not be perpendicular to k requires that we now depart from the analysis that was followed in Chapter 2 to discover the significance of the nonzero scalar product

$$\mathbf{k}\cdot\mathbf{E} \neq 0$$

To find what this dot product is equal to, we use Maxwell's equation to write

$$\mathbf{k}\times\mathbf{E} = \omega\mu_0\mathbf{H}$$

and then the cross product of this equation with k yields

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{E}) = \omega\mu_0\mathbf{k}\times\mathbf{H} = \omega\mu_0(-\omega\mathbf{D})$$

We can rewrite this equation by using the vector identity **(2A-3)**

$$\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) - \mathbf{E}(\mathbf{k}\cdot\mathbf{k}) = -\omega^2\mu_0\mathbf{D} \tag{13-8}$$

The propagation constant is given by

$$k^2 = \left(\frac{n\omega}{c}\right)^2$$

This allows **(13-8)** to be rewritten to yield an expression for the dot product $\mathbf{k}\cdot\mathbf{E}$

$$\frac{\mathbf{k}}{k^2}(\mathbf{k}\cdot\mathbf{E}) = \mathbf{E} - \frac{c^2\mu_0}{n^2}\mathbf{D} \tag{13-9}$$

$\mathbf{D}$ and $\mathbf{E}$ must satisfy **(13-9)** if they are to satisfy Maxwell's equations.

To allow a physical interpretation of **(13-9)**, we evaluate the components of this vector equation along the three principal directions. The *jth* component of the vector equation is

$$\frac{k_j}{k^2}(\mathbf{k}\cdot\mathbf{E}) = \left(1 - \frac{c^2\mu_0\epsilon_j}{n^2}\right)E_j \tag{13-10}$$

where the ϵ_j's are the principal dielectric constants. By multiplying both sides of the jth component by k_j, we obtain an eigenvalue equation. We have simplified the equation by making the substitution

$$c^2 = \frac{1}{\mu_0\epsilon_0}$$

$$\frac{k_j^2}{k^2\left(1 - \frac{\epsilon_j}{n^2\epsilon_0}\right)}(\mathbf{k}\cdot\mathbf{E}) = k_j E_j \tag{13-11}$$

Adding the three component equations of the form **(13-11)** together yields a new expression of the vector equation **(13-9)**

$$(\mathbf{k}\cdot\mathbf{E})\sum_{j=1}^{3}\frac{k_j^2}{k^2\left(1 - \frac{\epsilon_j}{n^2\epsilon_0}\right)} = \mathbf{k}\cdot\mathbf{E}$$

If $\mathbf{k}\cdot\mathbf{E} \neq 0$, as we implied might be the case, then we may divide both sides of the equation by the scalar product. We also remove n^2 from the summation to obtain

$$\sum_{j=1}^{3}\frac{k_j^2}{k^2\left(n^2 - \frac{\epsilon_j}{\epsilon_0}\right)} = \frac{1}{n^2} \tag{13-12}$$

A new parameter, which we will call *the principal refractive index*, is definite as

$$n_j^2 = \frac{\epsilon_j}{\epsilon_0} \tag{13-13}$$

Using the new definition, we can write **(13-12)** in the form called *Fresnel's equation*

$$\sum_{j=1}^{3}\frac{k_j^2}{k^2(n^2 - n_j^2)} = \frac{1}{n^2} \tag{13-14}$$

This equation allows the calculation of the index of diffraction for an arbitrary propagation direction. It appears to be cubic in n^2, but the equation is actually only quadratic in n^2 with two positive roots: n_1^2 and n_2^2. Thus, Fresnel's equation **(13-14)** states that for any propagation direction $\mathbf{k}$, there are, in general, two values of the refractive index (n_1 and n_2). If the solutions of **(13-14)** are substituted into **(13-10)**, two values of the electric field ($\mathbf{E}_1$

and $\mathbf{E}_2$) are obtained for the selected propagation direction. Finally, the two solutions for the electric field corresponding to n_1 and n_2 can be substituted into **(13-9)** to yield two values for the electric displacement ($\mathbf{D}_1$ and $\mathbf{D}_2$).

The two electric displacement vectors $\mathbf{D}_1$ and $\mathbf{D}_2$ associated with a selected propagation vector are perpendicular to one another, as will be demonstrated below. These vectors specify the polarizations of two light waves propagating at two different velocities, given by n_1 and n_2, in the anisotropic material. From Maxwell's equations, we have generated a theoretical explanation of the birefringent observations in any optically anisotropic material.

We can demonstrate that the two electric displacement vectors are orthogonal by first decomposing $\mathbf{E}$ into components parallel ($\mathbf{E}_\parallel$) and perpendicular ($\mathbf{E}_\perp$) to $\mathbf{k}$

$$\mathbf{E}_\parallel = \frac{\mathbf{k}}{k^2}(\mathbf{k}\cdot\mathbf{E}), \qquad \mathbf{E}_\perp = \mathbf{E} - \frac{\mathbf{k}}{k^2}(\mathbf{k}\cdot\mathbf{E}) \tag{13-15}$$

By rewriting **(13-9)**, we discover a relationship between $\mathbf{D}$ and $\mathbf{E}_\perp$

$$\mathbf{D} = \frac{k^2}{\mu_0\omega^2}\left[\mathbf{E} - \frac{\mathbf{k}}{k^2}(\mathbf{k}\cdot\mathbf{E})\right] = \frac{n^2}{\mu_0 c^2}\mathbf{E}_\perp \tag{13-16}$$

Since we have two indices of refraction, **(13-16)** produces two equations

$$\mathbf{D}_1 = \frac{n_1^2}{\mu_0 c^2}\mathbf{E}_{1\perp}, \qquad \mathbf{D}_2 = \frac{n_2^2}{\mu_0 c^2}\mathbf{E}_{2\perp} \tag{13-17}$$

To evaluate the scalar product $\mathbf{D}_1\cdot\mathbf{D}_2$, we first evaluate the scalar product

$$\mathbf{E}_2\cdot\mathbf{D}_1 = \sum_{i=1}^{3} E_{2i}D_{1i} = \sum_{i=1}^{3}\sum_{j=1}^{3} E_{2i}\epsilon_{ij}E_{1j}$$

Since the dielectric tensor must be symmetric, the scalar product between $\mathbf{E}$ and $\mathbf{D}$ is identical for either index of refraction

$$\sum_{i=1}^{3}\sum_{j=1}^{3} E_{2i}\epsilon_{ij}E_{1j} = \sum_{j=1}^{3}\sum_{i=1}^{3} E_{1j}\epsilon_{ji}E_{2i} = \sum_{j=1}^{3} E_{1j}D_{2j}$$

and we may write

$$\mathbf{E}_2\cdot\mathbf{D}_1 = \mathbf{E}_1\cdot\mathbf{D}_2 \tag{13-18}$$

The electric field can be written as the sum of the components parallel and perpendicular to $\mathbf{k}$

$$(\mathbf{E}_{2\perp} + \mathbf{E}_{2\parallel})\cdot\mathbf{D}_1 = (\mathbf{E}_{1\perp} + E_{1\parallel})\cdot\mathbf{D}_2$$

From the definition in **(13-15)**, we know that $\mathbf{E}_\parallel\cdot\mathbf{E}_\perp = 0$ for all i and j. We combine this fact with **(13-17)** to write

$$(n_1^2 - n_2^2)(\mathbf{E}_{2\perp}\cdot\mathbf{E}_{1\perp}) = 0$$

If $n_1^2 \neq n_2^2$, then

$$\mathbf{E}_{2\perp}\cdot\mathbf{E}_{1\perp} = 0$$

and **(13-17)** allows us to write

$$\mathbf{D}_1\cdot\mathbf{D}_2 = \frac{n_1^2 n_2^2}{(\mu_0 c^2)^2}(\mathbf{E}_{1\perp}\cdot\mathbf{E}_{2\perp}) = 0 \tag{13-19}$$

Appendix 13-A

TENSORS

In this appendix, we will define a tensor. A geometrical construction of use in determining the properties of a symmetric, second-order tensor will be discussed. Second-order symmetric tensors for each of the crystal symmetry classes will also be presented.

A tensor is an abstract mathematical object that has specific components in every coordinate system and undergoes a well-defined transformation when the coordinate system undergoes a transformation, say, a rotation. We will define below several tensors that are of use in optics.

Scalar

Scalars are quantities that are not connected with direction, for example, charge. We can completely specify the value of a scalar by assigning it a single number. A scalar is a tensor of rank zero.

Vector

Vectors are quantities that must be defined with reference to a direction, for example, the electric field. To specify a vector's value at a point, we must give both its magnitude and direction or, equivalently, we may specify the vector's components along three mutually perpendicular directions

$$\mathbf{E} = \{E_1, E_2, E_3\}$$

A vector is called a tensor of rank one.

Second Rank Tensor

Tensors of second rank relate two general vectors. If we wish to relate displacement **D** and electric field **E**, we introduce the dielectric tensor $\bar{\epsilon}$ such that

$$D_i = \sum_{j=1}^{3} \epsilon_{ij} E_j \qquad (13A\text{-}1)$$

The constants ϵ_{ij} are the components of the tensor $\bar{\epsilon}$ and can be given the following physical meaning. Suppose the electric field is in the x direction

$$\mathbf{E} = \{E_1, 0, 0\}$$

Then, **D** would have components, according to **(13A-1)**, along all three axes

$$\mathbf{D} = \{\epsilon_{11}E_1, \epsilon_{21}E_1, \epsilon_{31}E_1\}$$

The component ϵ_{21} gives the component of **D** in the y direction due to the electric field in the x direction.

In matrix notation, **(13A-1)** would be written

$$(D_1 \quad D_2 \quad D_3) = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \tag{13A-2}$$

The 3×3 matrix of ϵ's is the tensor of the second rank, denoted by $\bar{\epsilon}$. In general, there are nine components in a second-rank tensor.

Higher-Rank Tensor

A tensor of third rank relates a vector **A** to two vectors **B** and **C**. In general, each component of **A** will depend on all the components of **B** and **C**.

$$A_i = \sum_{j=1}^{3} \sum_{k=1}^{3} S_{ijk} B_j C_k$$

The S_{ijk} are components of a third-rank tensor. In general, an Nth-rank tensor relates one vector to $(N-1)$ other vectors.

Coordinate Transformation

The components of the tensor will change if a coordinate transformation is made. In fact, the way in which a tensor changes, under a coordinate transformation, is part of its definition. The transformation from one rectangular coordinate system to another is defined by the equation

$$x_i' = \sum_{j=1}^{3} \gamma_{ij} x_j$$

To transform a second-rank tensor between the two coordinate systems defined by this transformation, we use the equation

$$\epsilon_{ij}' = \sum_{k=1}^{3} \sum_{\ell=1}^{3} \gamma_{ik} \gamma_{j\ell} \epsilon_{k\ell} \tag{13A-3}$$

Third-rank tensors would transform according to the equation

$$S_{ijk}' = \sum_{\ell=1}^{3} \sum_{m=1}^{3} \sum_{n=1}^{3} \gamma_{i\ell} \gamma_{jm} \gamma_{kn} S_{\ell mn}$$

The physical property that the tensor represents (in the example of a second-rank tensor, the dielectric constant) should be uniquely defined by the tensor, independent of the coordinate system. The tensor selected to represent the physical property is the simplest matrix that can be constructed.

If the tensor is symmetric, that is, if $\epsilon_{kj} = \epsilon_{jk}$, then by proper choice of the coordinate system, we can write the second-rank tensor as a diagonal matrix, reducing the number of constants from six to three. The three elements of the diagonal tensor are called the principal elements of the tensor and are the quantities used to characterize the physical property described by the tensor.

Geometrical Representation

Our understanding of symmetrical tensors of the second rank can be aided by the use of a geometric construction. Equation **(13A-1)**, in its most general

form, is a second-degree equation that, for an arbitrary set of unit vectors, is written

$$\sum_{i=1}^{3}\sum_{j=1}^{3} S_{ij}x_i x_j = 1$$

If the tensor is symmetric, $S_{ij} = S_{ji}$, then the second-degree equation can be written in expanded form as

$$S_{11}x_1^2 + S_{22}x_2^2 + S_{33}x_3^2 + 2(S_{23}x_2x_3 + S_{31}x_3x_1 + S_{12}x_1x_2) = 1 \qquad (13A\text{-}4)$$

This is the general equation of a quadric (a second-degree surface or conicoid). The surface described by this equation may be an ellipsoid or a hyperboloid.

The coefficients S_{ij} of **(13A-4)** transform according to the same equation as the components of a symmetrical tensor of second rank **(13A-3)**, and thus, we can say that S is a tensor. The surfaces in Figure 13A-1 are aligned with the principal axes; therefore, the equation for the surfaces shown is the simplified form

$$S_1x_1^2 + S_2x_2^2 + S_3x_3^2 = 1 \qquad (13A\text{-}5)$$

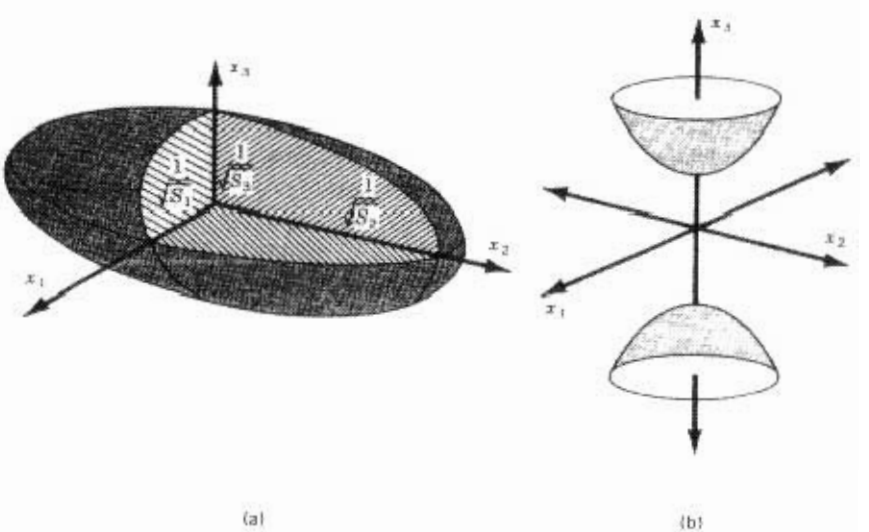
The standard equation of the quadric is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (13A\text{-}6)$$

Comparing **(13A-5)** with **(13A-6)**, we see that the semiaxes of the ellipsoid in Figure 13A-1 are equal to the principal elements of the tensor **S**.

$$a = \frac{1}{\sqrt{S_1}}, \qquad b = \frac{1}{\sqrt{S_2}}, \qquad c = \frac{1}{\sqrt{S_3}}$$

The major use of the geometrical representation of Figure 13A-1 is in providing an understanding of how symmetric tensors of second rank change with coordinate transformations.



(a) (b)

FIGURE 13A-1. The quadric surface described by **(13A-4)**. (a) An ellipsoid, (b) a hyperboloid of two sheets.

If the symmetrical tensor S is referred to an arbitrary set of axes, six independent components (S_{11}, S_{22}, S_{33}, S_{23}, S_{31}, and S_{12}) are needed to describe the surface. Naturally, we would try, when possible, to refer the tensor to its principal axes, the surface's symmetry axes, so that only the three principal components are needed.

Crystal Symmetry

Another way that the number of required components of a tensor is reduced is when the environment requires that the tensor obey certain symmetry operations. For example, the dielectric constant of a crystal must mirror the symmetry exhibited by the crystal. The crystallographic symmetry classes can be divided into three groups according to the number of symmetry axes present in the crystal. The groups, and the crystal classes that are members of the groups, are listed in Table 13A.1.

TABLE 13A.1 Crystal Classes and Their Second-Rank Tensor

Group	Quadric Surface	Orientation of Quadric	Crystal Class	Tensor	Number of Coefficients	Optical Type
Three equivalent orthogonal directions	Sphere	Unimportant	Cubic, three 4-fold axes	$\begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S \end{bmatrix}$	1	Isotropic
Two equivalent orthogonal directions in a plane normal to the crystal's symmetry axis	Spheroid	One principal axis parallel to the crystal symmetry axis	Trigonal, one 3-fold symmetry axis	$\begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_1 & 0 \\ 0 & 0 & S_2 \end{bmatrix}$	2	Uniaxial
			Tetragonal, one 4-fold symmetry axis			
			Hexagonal, one 6-fold symmetry axis			
No equivalent directions	General Quadric	Not specified	Triclinic, no symmetry (at most, a center of symmetry)	$\begin{bmatrix} S_{11} & S_{12} & S_{31} \\ S_{12} & S_{22} & S_{23} \\ S_{31} & S_{23} & S_{33} \end{bmatrix}$	6	Biaxial
		Principal axis parallel to symmetry axis	Monoclinic one 2-fold symmetry axis	$\begin{bmatrix} S_1 & 0 & S_{31} \\ 0 & S_2 & 0 \\ S_{31} & 0 & S_3 \end{bmatrix}$	4	
		Principal axes aligned with symmetry axes	Orthorombic, three perpendicular 2-fold axes of symmetry	$\begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix}$	3	