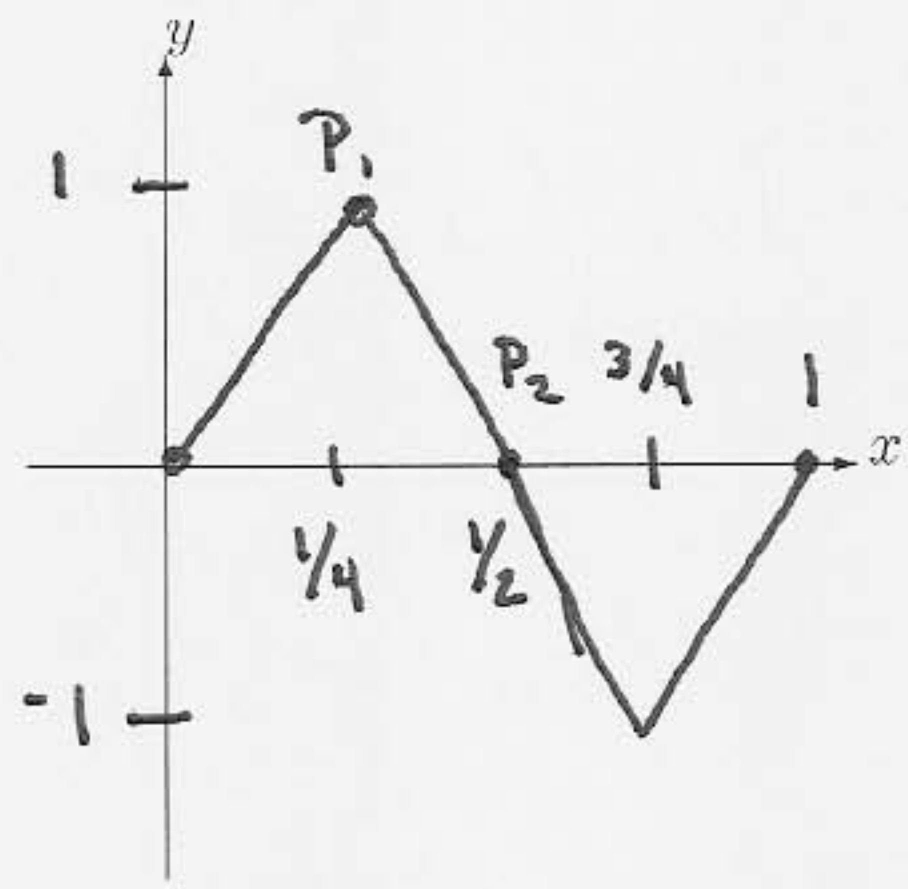
MATH 348 - April 14, 2008
Exam II - 50 Points - 50 minutes

NAME: Key
SECTION:

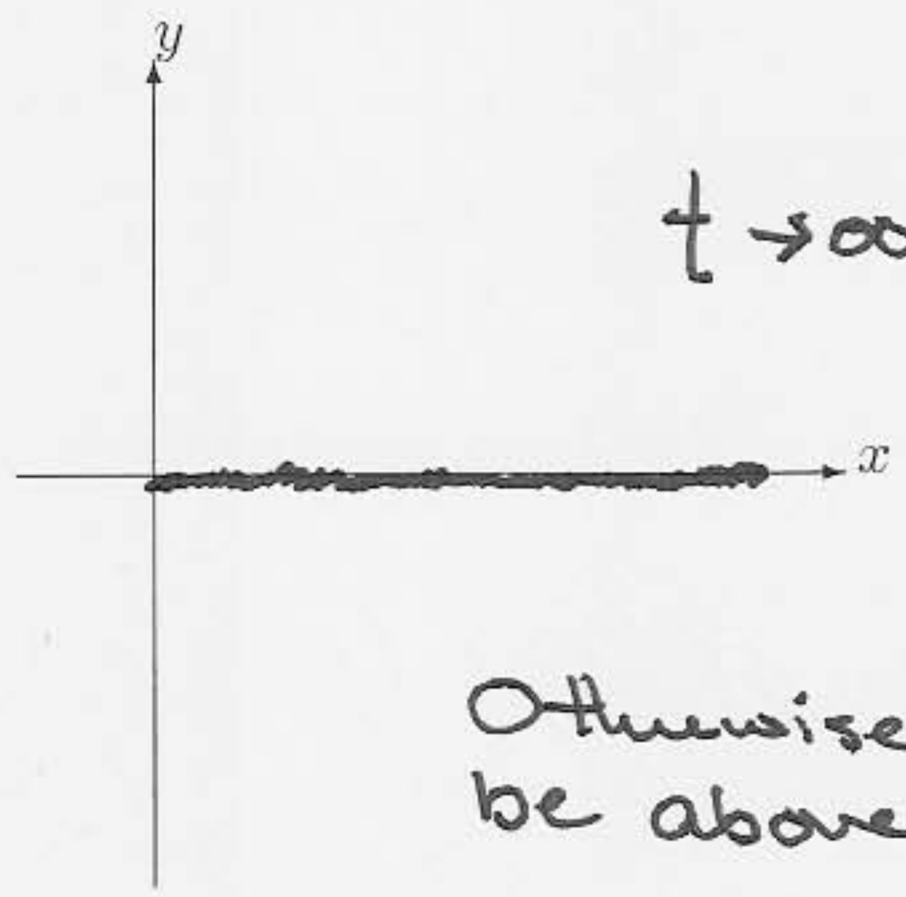
In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points) Conceptual Questions.
   Suppose that we have the graph of the initial condition $u(x,0) = f(x)$:



   (a) Assume that f is the initial temperature for a homogenous heat problem with boundary conditions, $u_x(0,t) = 0$, $u_x(1,t) = 0$. Describe the physical meaning of these boundary conditions and graph the approximate temperature profile for $t \to \infty$.

$u_x(0,t) = u_x(1,t) = 0$
Implies that the Endpoints of the object are insulated from the universe (Heat bath)



Assuming the area of both triangles is the same.

Otherwise the line would be above or below axis.

   (b) Assume that f is the initial displacement of an elastic string modeled by the homogenous wave equation subject to boundary conditions $u(0,t) = 0$, $u(1,t) = 0$. Describe the physical meaning of these boundary conditions and describe time-dynamics of the points, P_1 and P_2, on this elastic string assuming that the string has no initial velocity.

These B.C. Imply that the String is fixed at both End points.

P_1 will oscillate between [-1,1] taking on all points in between.

P_2 will stay fixed for all time.

2. (10 Points)

(a) Show that $u(x,t) = \dfrac{1}{x-ct}$ is a solution to the one-dimensional wave equation.

$$\frac{\partial u}{\partial t} = \frac{-1}{(x-ct)^2} \cdot -c \Rightarrow u_{tt} = c^2 \cdot \frac{2}{(x-ct)^3}$$

$$u_{xx} = \frac{2}{(x-ct)^3} \Rightarrow u_{tt} = c^2 u_{xx}$$

(b) Show that $u(x,y) = \ln(x^2+y^2)$ is a solution to $u_{xx} + u_{yy} = 0$.

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x$$

$$\Rightarrow u_{xx} = \frac{2}{x^2+y^2} - \frac{2x \cdot 2x}{(x^2+y^2)^2}$$

$$\Rightarrow u_{xx} + u_{yy} = \frac{4}{x^2+y^2} - \frac{4(x^2+y^2)}{(x^2+y^2)^2} = 0$$ ✓

3. (10 Points) Given,

$$\frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

Using separation of variables, $u(x,t) = F(x)G(t)$, and find two ODE's associated with the PDE. Determine the general solution to each of the ODE's, assuming the separation constant $k = 3$.

$$(1) \Leftrightarrow \frac{G' + G}{G} = \frac{F''}{F} = 3$$

$$\Rightarrow G' = 3G - G = 2G, \quad F'' + 3F = 0$$

$$G(t) = C_1 e^{2t} \qquad \Rightarrow F(x) = C_2 \sin(\sqrt{3}t) + C_3 \cos(\sqrt{3}t)$$

4. (10 Points) Suppose that the one-dimensional wave equation gives rise to the boundary value problem,

$$F''(x) + kF(x) = 0, \;\; k \geq 0, \tag{2}$$

$$F'(0) = 0, \quad F'(1) = 0. \tag{3}$$

(a) Explain the physical interpretation of the boundary conditions (3).

For the wave Equ this implies the endpoints are flat.

(b) Find all solutions to the the BVP (2)-(3).

$$F_n(x) = \cos\left(\frac{n\pi}{L=1}x\right), \; \sqrt{k_n} = \frac{n\pi}{L}, \quad \text{See HW\#9}$$

$$F_0(x) = C \in \mathbb{R}, \quad \sqrt{k} = 0$$

5. (10 Points) Suppose that we know that,

$$\begin{aligned} G_n(t) &= B_n e^{-k_n c^2 t}, \;\; B_n \in \mathbb{R}, & (4)\\ F_n(x) &= \cos(k_n x), \;\; k_n = n\pi, \;\; n = 0, 1, 2, \cdots, & (5) \end{aligned}$$

are the temporal and spatial solutions to some heat equation. Assuming that $u(x, 0) = f(x)$:

(a) Write down the general solution to the PDE.

$$u(x,t) = \sum_{n=1}^{\infty} F_n(x) G_n(t) = \sum_{n=0}^{\infty} B_n e^{-k_n c^2 t} \cos(k_n x)$$

(b) Solve for any unknown constants in terms of $f(x)$. $n \geq 1$

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} B_n \cos(k_n x) \Rightarrow B_n = \frac{2}{L=1} \int_0^1 f(x)\cos(k_n x)\,dx$$

$$n=0 \quad B_0 = \cancel{2}\frac{1}{L=1}\int_0^1 f(x)\,dx$$

(c) What is long term behavior of the temperature of this one-dimensional object?

$$\lim_{t \to \infty} u(x,t) = B_0$$