MATH 332 - Linear Algebra July 14, 2009

Homework 6, Summer 2009 **Due**: July 20, 2009

Vector Spaces - Subspaces - Bases and Coordiantes - Classical Matrix Spaces - Abstract Vector Spaces

1. (a) Verify that the set of all n-times continuously differentiable functions on $[a, b]$, which satisfies the homogeneous linear ordinary differential equation $L[y] = 0$,

$$V = \left\{ y \in C^{(n)}[a,b] : L[y] = a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0(t)y = 0, \text{ where } a_0, \ldots, a_n \in C[a,b] \right\},$$

is a vector space under addition of functions and scalar multiplication.

(b) Prove that if H is the set of all polynomials up to degree n, such that $p(0) = 0$, then H is a subspace of $\mathbb{P}_n$.

(c) Prove that if $H = \{f \in C[a,b] : f(a) = f(b)\}$, then H is a subspace of $C[a,b]$.

2. The standard basis for $\mathbb{R}^2$ are the column vectors, $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbf{I}_{2\times2}$. In class we looked at the basis $\mathfrak{B} = \{[1,1]^{\mathrm{T}}, [-1,1]^{\mathrm{T}}\}$. This basis is rotated $\frac{\pi}{4}$ radians counter-clockwise from the standard basis and does not preserve the notion of length from the standard coordinate system.

(a) Determine a basis for $\mathbb{R}^2$, which is rotated $\frac{\pi}{4}$ radians counter-clockwise from the standard basis and preserves the unit length associated with the standard basis.

(b) Show that, for this basis, the change-of-coordinates matrix $\mathbf{P}_{\mathfrak{B}}$ is such that, $\mathbf{P}_{\mathfrak{B}}\mathbf{P}_{\mathfrak{B}}^{\mathrm{T}} = \mathbf{P}_{\mathfrak{B}}^{\mathrm{T}}\mathbf{P}_{\mathfrak{B}} = \mathbf{I}_{2\times2}$.

(c) Given that $[\mathbf{x}_1]_{\mathfrak{B}} = [\sqrt{2}, \sqrt{2}]^{\mathrm{T}}$ determine $\mathbf{x}_1$ and given that $\mathbf{x}_2 = \left[\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right]^{\mathrm{T}}$ determine $[\mathbf{x}_2]_{\mathfrak{B}}$. Calculate the magnitude of both of the vectors previously calculated.

3. Given,

$$\mathbf{A} = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

(a) Is $\mathbf{w}$ in the column space of $\mathbf{A}$? That is, does $\mathbf{w} \in \text{Col } \mathbf{A}$?

(b) Is $\mathbf{w}$ in the null space of $\mathbf{A}$? That is, does $\mathbf{w} \in \text{Nul } \mathbf{A}$?

4. Given,

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}.$$

(a) Determine a basis and the dimension of Nul $\mathbf{A}$.

(b) Determine a basis and the dimension of Col $\mathbf{A}$.

(c) Determine a basis and the dimension of Row $\mathbf{A}$.

5. The Hermite polynomials are a sequence of orthogonal polynomials, which arise in probability, combinatorics and physics.[1] The first four polynomials in this sequence are given as,

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = -2 + 4x^2, \quad H_3(x) = -12x + 8x^3, \quad x \in (-\infty, \infty).$$

(a) Show that $\mathfrak{B} = \{1, 2x, -2 + 4x, -12x + 8x^3\}$ is a basis for $\mathbb{P}_3$.
Hint: Determine the coordinate vectors of the Hermite polynomials relative to the standard basis.

(b) Let $\mathbf{p}(x) = 7 - 12x - 8x^2 + 12x^3$. Find the coordinate vector of $\mathbf{p}$ relative to $\mathfrak{B}$.
Hint: Determine $\{c_0, c_1, c_2, c_3\}$ such that $\mathbf{p}(x) = \sum_{i=0}^{3} c_i H_i(\mathrm{x})$.

[1] In physics these polynomials manifest as the spatial solutions to Schrödinger's wave equation under a harmonic potential, which evolves the probability distribution of a quantum mechanical particle near an energy minimum. As it turns out there are infinitely-many Hermite polynomials and consequently one can show that this particle has infinitely-many allowed quantized energy levels, which are evenly spaced.