

8

Phase matching in frequency mixing  
 Non-collinear beam phase matching  
 Power conservation, dimensionless scaled  
 equations

## Gain coefficient

- Gain can be very high:

$$2\kappa = 2 \times 2d_{\text{eff}} \sqrt{\frac{\omega_1 \omega_2}{c^2 n_1 n_2}} |A_3| = 4d_{\text{eff}} \sqrt{\frac{\omega_1 \omega_2}{c^2 n_1 n_2}} \sqrt{\frac{I_3}{2n_3 \epsilon_0 c}} \quad 1V/m = \sqrt{377W/m^2}$$

$$1W/m^2 = 10000W/cm^2 = \sqrt{\frac{1}{377}} V/m$$

- For BBO

$$2\kappa \approx 4d_{\text{eff}} \frac{2\pi}{n\lambda_p/2} \sqrt{\frac{I_3}{n\epsilon_0 c/2}} = \frac{16\pi}{n} \frac{1}{\lambda_p} d_{\text{eff}} \sqrt{\frac{I_3}{2n\epsilon_0 c}}$$

$$d_{\text{eff}} = 2pm/V$$

$$n = 1.6$$

$$\kappa \approx 7 \times 10^{-4} \frac{\sqrt{I_3 (W/cm^2)}}{\lambda_p (\mu m)} / cm$$

$$\text{Exp}[14] = 10^6$$

For this single pass gain and L = 5mm, need

$$2\kappa = 14/L = 28/cm$$

Need about  $I_3 = 2 \times 10^9 W/cm^2$

## Phase matching for parametric mixing

$$\Delta k = k_1 + k_2 - k_3$$

$$= \frac{1}{c}(\omega_1 n_1 + \omega_2 n_2 - \omega_3 n_3)$$

Note: for wave mixing, phase matching isn't just matching phase velocities or ref. indices. For SHG:

$$\Delta k = 2k_1 - k_2 = \frac{2\omega_1}{c}(n_1 - n_2)$$

- For negative uniaxial:  $n_e$  is lowest, so place  $\omega_3$  as  $e$ -wave
- Type I:  $\omega_1$  and  $\omega_2$  parallel polarization, along  $o$ -direction
  - Generally broader bandwidth for OPA
- Type II: one of lower frequencies is along  $e$ -direction
  - can separate signal and idler with polarizer
- Quasi-phasematching (2.4), periodic poling
- Non-collinear phase matching: momentum conservation is a vector relation

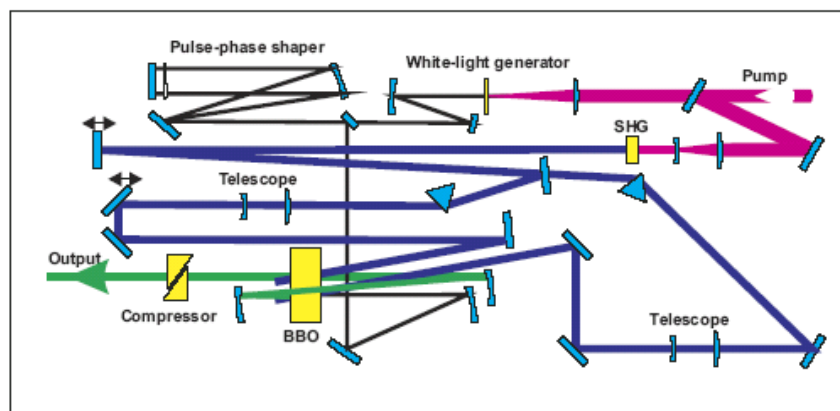
## An ultrafast noncollinear OPA (NOPA)

Continuum generates an arbitrary-color seed pulse.

- Pump with blue
- 2 stage, 1 crystal
- Noncollinear pump for broadband phase matching

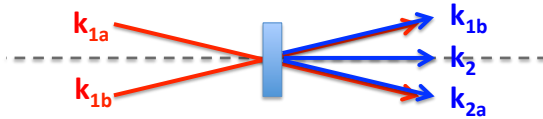


*TOPAS-white*



## Non-collinear phase matching: SHG=SFG

- We can cross two beams of the same frequency at a crystal



- SHG along the input directions, SFG down the middle
  - For SFG: signal takes one photon from each beam
- Non-collinear phase matching: momentum conservation is a vector relation  $\Delta \mathbf{k} = \mathbf{k}_{1a} + \mathbf{k}_{1b} - \mathbf{k}_2$ 
  - Place z-axis along centerline = direction of sum frequency

$$\hat{z} \Delta k = \hat{z} \left( \frac{\omega_1}{c} n_1 \cos(\theta) + \frac{\omega_1}{c} n_1 \cos(-\theta) - \frac{\omega_2}{c} n_2 \right)$$

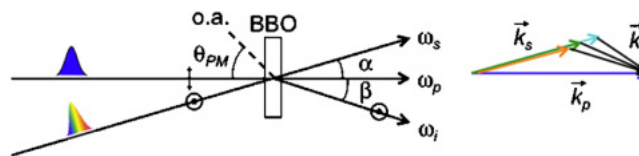
$$\Delta k = \frac{2\omega_1}{c} (n_1 \cos(\theta) - n_2)$$

Phase matching condition is modified by crossing angle

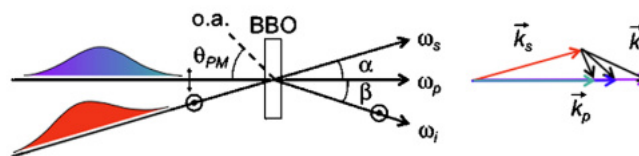
## Noncollinear OPA (NOPA)

- Narrowband pump, broadband, chirped seed
- Choose crossing angle so that phase matching is satisfied for a wide range of frequency

(a) fs NOPA



(b) ps NOPA



## Power conservation in frequency mixing

- Manley-Rowe relations (2.5)

$$\frac{d}{dz} \left( \frac{I_1}{\omega_1} \right) = \frac{d}{dz} \left( \frac{I_2}{\omega_2} \right) = - \frac{d}{dz} \left( \frac{I_3}{\omega_3} \right)$$

- Can use these to reduce the number of coupled equations
- Helpful to understand saturated conversion limits in parametric processes:
  - $1/\hbar\omega = \text{photon flux}$
  - For every photon taken from wave 3, we add one to waves 1 and 2
  - Example: can avoid back conversion in sum frequency generation only if there isn't any light left in both input beams

## Scaling the NL equations

- Using scaled variables can simplify the NL equations and highlight the characteristic parameters

- SHG example  $\omega_2 = 2\omega_1$   $\Delta k = 2k_1 - k_2$  2.7.10, 2.7.11

$$\frac{dA_1}{dz} = \frac{2i d_{\text{eff}} \omega_1^2}{k_1 c^2} A_2 A_1^* e^{-i\Delta k z} \quad \frac{dA_2}{dz} = \frac{i d_{\text{eff}} \omega_2^2}{k_2 c^2} A_1^2 e^{+i\Delta k z}$$

- Scale the fields to the total intensity

– Intensity of  $j^{\text{th}}$  field  $I_j = 2n_j \epsilon_0 c |A_j|^2$   $a_j = u_j e^{i\theta_j}$

$$|a_j|^2 = u_j^2 = \frac{I_j}{I} = \frac{2n_j \epsilon_0 c}{I} |A_j|^2$$

$a_j$  is scaled, complex field

$$A_j = \sqrt{\frac{I}{2n_j \epsilon_0 c}} u_j e^{i\theta_j} = \sqrt{\frac{I}{2n_j \epsilon_0 c}} a_j$$

## Writing dimensionless equations

- Rewrite equations with new variables

$$A_1 = \sqrt{\frac{I}{2n_1\epsilon_0 c}} a_1 \quad A_2 = \sqrt{\frac{I}{2n_2\epsilon_0 c}} a_2$$

– Total power is conserved:  $|a_1|^2 + |a_2|^2 = u_1^2 + u_2^2 = 1$

$$\frac{dA_1}{dz} = \frac{2id_{\text{eff}}\omega_1^2}{k_1c^2} A_2 A_1^* e^{-i\Delta kz} \quad \rightarrow \frac{da_1}{dz} = \frac{2id_{\text{eff}}\omega_1^2}{k_1c^2} \sqrt{\frac{I}{2n_2\epsilon_0 c}} a_2 a_1^* e^{-i\Delta kz}$$

$$\frac{dA_2}{dz} = \frac{id_{\text{eff}}\omega_2^2}{k_2c^2} A_1^2 e^{+i\Delta kz} \quad \rightarrow \sqrt{\frac{I}{2n_2\epsilon_0 c}} \frac{da_2}{dz} = i \frac{d_{\text{eff}}2\omega_1}{n_2c} \frac{I}{2n_1\epsilon_0 c} a_1^2 e^{+i\Delta kz}$$

$$\frac{da_2}{dz} = i \frac{2d_{\text{eff}}\omega_1}{n_1c} \sqrt{\frac{I}{2n_2\epsilon_0 c}} a_1^2 e^{+i\Delta kz}$$

## Final form of scaled equations for SHG

$$\frac{da_1}{dz} = \frac{2id_{\text{eff}}\omega_1^2}{k_1c^2} \sqrt{\frac{I}{2n_2\epsilon_0 c}} a_2 a_1^* e^{-i\Delta kz} \quad \frac{da_2}{dz} = i \frac{2d_{\text{eff}}\omega_1}{n_1c} \sqrt{\frac{I}{2n_2\epsilon_0 c}} a_1^2 e^{+i\Delta kz}$$

$$|a_1|^2 + |a_2|^2 = u_1^2 + u_2^2 = 1$$

- From 2<sup>nd</sup> equation, identify scale length for signal growth

– Distance variable  $l = \frac{c}{2\omega_1 d_{\text{eff}}} \sqrt{\frac{2n_1^2 n_2 \epsilon_0 c}{I}}$

- Rewrite equations

$$\frac{da_1}{dz} = i \frac{1}{l} a_2 a_1^* e^{-i\Delta kz} \quad \frac{da_2}{dz} = i \frac{1}{l} a_1^2 e^{+i\Delta kz} \quad \xi = z/l$$

- Define dimensionless distance variable  $\Delta k z = \Delta k l \xi \equiv \Delta s \xi$

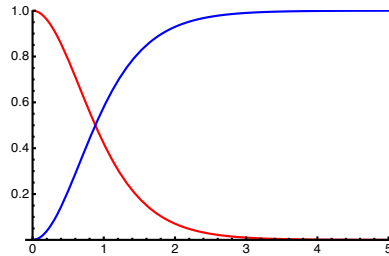
$$\frac{da_1}{d\xi} = i a_2 a_1^* e^{-i\Delta s \xi}$$

$$\frac{da_2}{d\xi} = i a_1^2 e^{+i\Delta s \xi}$$

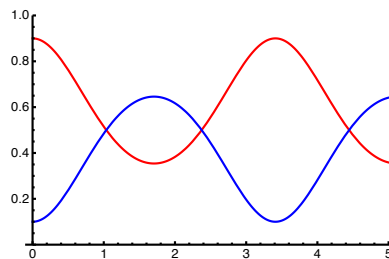
- $l$  is the characteristic distance for energy exchange (saturation and back-conversion)
- If there is no SH present at start, full conversion

## Saturated SHG conversion

- No seed SH



- With seed

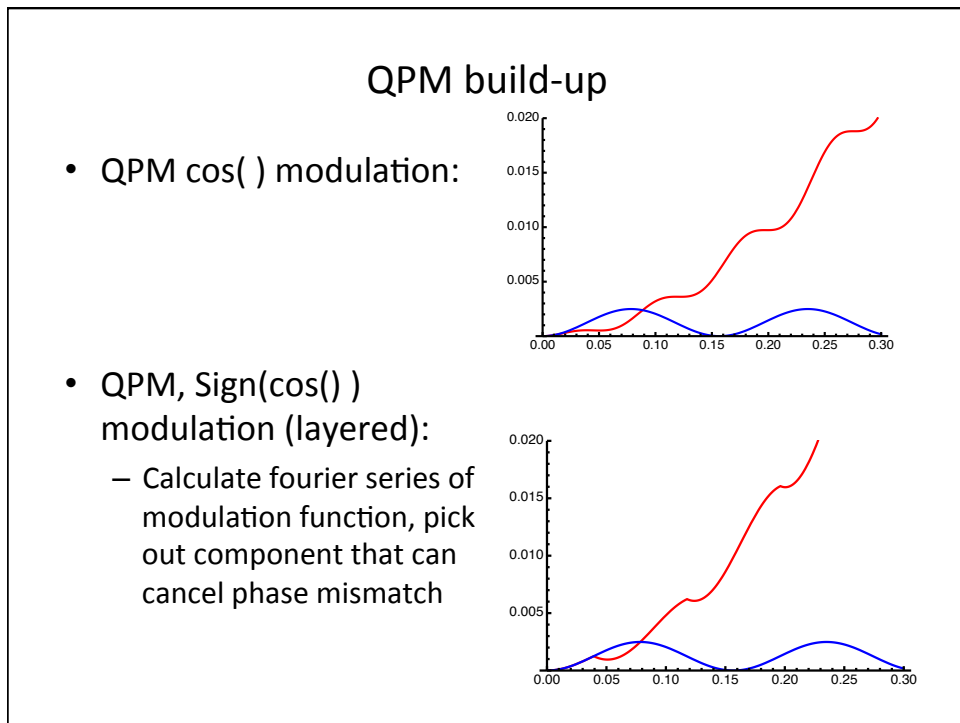
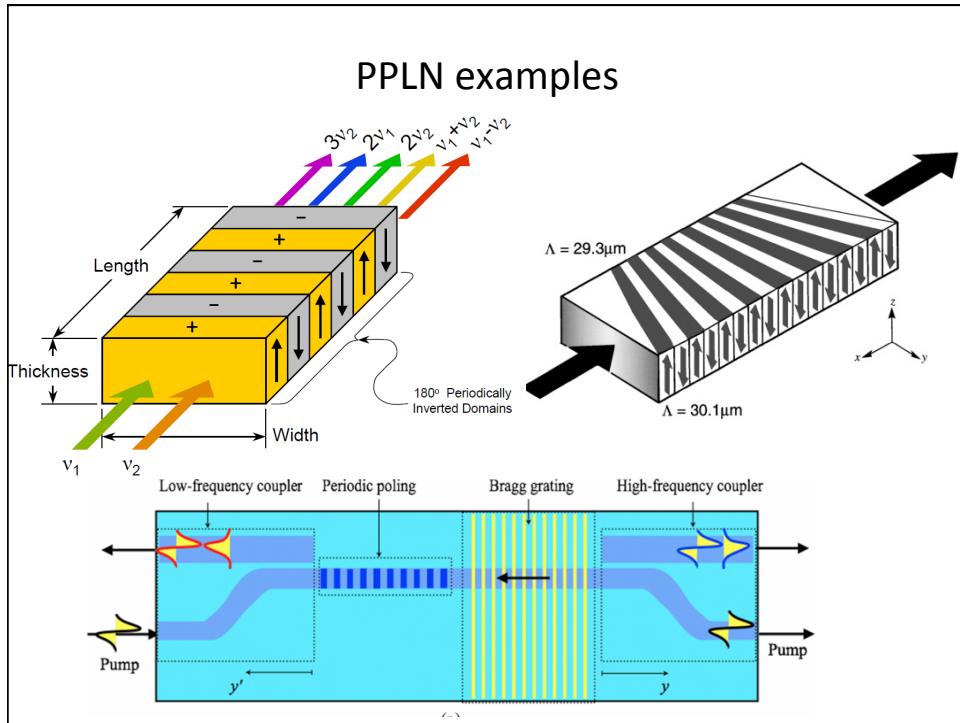


## Quasi-phase matching

- Some materials don't support birefringent phase matching
  - LiNbO<sub>3</sub> has a strong NL coefficient but in the same vector direction as the input polarization
  - Isotropic materials, e.g. gas or liquid
- Structuring the medium can allow build-up of NL signal without complete phase matching

$$\frac{dA_3}{dz} = \frac{2i d_{eff} \omega_3^2}{k_3 c^2} A_1 A_2 e^{+i\Delta k z} \quad d_{eff}(z) = d_0 \cos Kz$$

$$\begin{aligned} \frac{dA_3}{dz} &= i \frac{d_0 \omega_3^2}{k_3 c^2} A_1 A_2 (e^{+iKz} + e^{-iKz}) e^{+i\Delta k z} \\ &= i \frac{d_0 \omega_3^2}{k_3 c^2} A_1 A_2 (e^{+i(K+\Delta k)z} + e^{-i(K-\Delta k)z}) \end{aligned} \quad \begin{array}{l} \text{if } K \pm \Delta k = 0, \\ \text{signal can build up} \end{array}$$



### 3D propagation

$$\nabla^2 \mathbf{E}_j - \frac{n_j^2}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E}_j = \frac{\partial^2}{\partial z^2} \mathbf{E}_j + \nabla_{\perp}^2 \mathbf{E}_j - \frac{n_j^2}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E}_j = \frac{1}{\epsilon_0 c^2} \frac{\partial^2}{\partial t^2} \mathbf{P}_j^{NL}$$

- **Notes:**
  - RHS is source term
  - All linear propagation effects are included in LHS: diffraction, interference, focusing...
- So far, we've assumed plane waves where transverse derivatives are zero.
- Counter examples:
  - Gaussian beams (including high-order)
  - Waveguides
  - Arbitrary propagation
- Often determine solutions to linear equation (e.g. Gaussian beams, waveguide modes), then express fields in terms of those solutions.

$$\nabla_{\perp}^2 = \partial_x^2 + \partial_y^2$$

$$\nabla_{\perp}^2 = \frac{1}{r} \partial_r (r \partial_r) + \frac{1}{r^2} \partial_{\phi}^2$$

### Slowly-varying envelope approximation

- Assume waves are forward-propagating:

$$\mathbf{E}_j(\mathbf{r}, t) = \mathbf{A}_j(\mathbf{r}) e^{i(k_j z - \omega_j t)} + \text{c.c.}$$

$$\mathbf{P}_j(\mathbf{r}, t) = \mathbf{p}_j(\mathbf{r}) e^{i(k_j z - \omega_j t)} + \text{c.c.}$$

$$\frac{\partial^2}{\partial z^2} \mathbf{A}_j + 2ik_j \frac{\partial}{\partial z} \mathbf{A}_j - k_j^2 \mathbf{A}_j + \nabla_{\perp}^2 \mathbf{A}_j + \frac{n_j^2 \omega_j^2}{c^2} \mathbf{A}_j = -\frac{\omega_j^2}{\epsilon_0 c^2} \mathbf{p}_j e^{i\Delta k z}$$

- Fast oscillating carrier terms cancel (blue)
- Slowly-varying envelope: compare red terms
  - Drop 2<sup>nd</sup> order deriv if  $\frac{2\pi}{\lambda_j} \frac{1}{L} A_j \gg \frac{1}{L^2} A_j$
  - Ignoring any counterpropagating waves



## Diffractive propagation

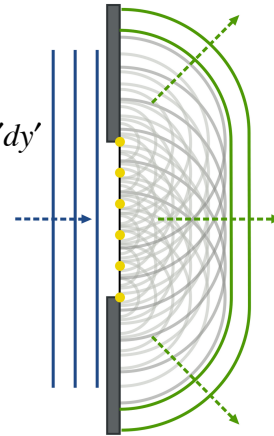
- Huygens' principle:
  - Represent a plane wave as a superposition of source points emitting spherical waves
- Integral representation:

$$E(x, y, z) = \frac{i}{\lambda} \iint E(x', y', z') \frac{\exp[-ik|\mathbf{r} - \mathbf{r}'|]}{|\mathbf{r} - \mathbf{r}'|} \cos\theta dx' dy'$$

Field at input plane

Spherical wavelet

Inclination factor



This is essentially a convolution of the complex input field with the spherical wavelets, which are the Green's function for the wave equation

## Paraxial approximations

- For **rays**, paraxial = small angle to optical axis
  - Ray slope:  $\tan\theta \approx \theta$
- For **spherical waves** where power is directed forward:

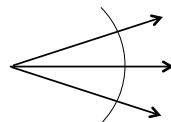
$$e^{ikr} = \exp\left[ik\sqrt{x^2 + y^2 + z^2}\right]$$

$$k\sqrt{x^2 + y^2 + z^2} = kz\sqrt{1 + \frac{x^2 + y^2}{z^2}} \approx kz\left(1 + \frac{x^2 + y^2}{2z^2}\right) \quad \text{Expanding to 1st order}$$

$$e^{i(kr - \omega t)} \rightarrow e^{ikz} \exp\left[i\left(k\frac{x^2 + y^2}{2z} - \omega t\right)\right] \quad z \text{ is radius of curvature}$$

Wavefront = surface of constant phase  
For  $x, y > 0$ ,  $t$  must increase.  
Wave is diverging:

$$k\frac{x^2 + y^2}{2z} = \omega t$$



## Fresnel diffraction integral

- Fresnel approximation (near field)
  - Expand the spherical wave in paraxial approximation (in exponential)
  - Let denominator be  $|\mathbf{r}-\mathbf{r}'| \sim z-z' = L \quad \cos\theta \simeq 1$
  - Input field:  $E(x',y',z') = u(x',y',z')e^{-ik(z-z')}$

$$u(x,y,z) = \frac{i}{\lambda L} \iint u(x',y',z') \exp\left[-ik \frac{(x-x')^2 + (y-y')^2}{2L}\right] dx' dy'$$

$$u(x,y,z) = \frac{i}{\lambda L} e^{-ik \frac{x^2+y^2}{2L}} \iint u(x',y',z') e^{-ik \frac{x'^2+y'^2}{2L}} e^{-i \frac{k}{L}(xx'+yy')} dx' dy'$$

## Gaussian beam solution to wave equation

- Use Fresnel integral to propagate a Gaussian beam

$$U(x,y,z) = \frac{1}{i\lambda z} e^{ikz} e^{ik \frac{x^2+y^2}{2z}} \iint \left[ e^{-\frac{x'^2+y'^2}{w^2}} e^{ik \frac{x'^2+y'^2}{2z}} \right] e^{i2\pi(f_x x' + f_y y')} dx' dy'$$

- Combine quadratic terms in exponent:

$$\left(-\frac{1}{w^2} + i \frac{k}{2z}\right) = i \frac{k}{2} \left(\frac{1}{z} + i \frac{2}{kw^2}\right) = i \frac{k}{2q}$$

Sign convention for  $-i\omega t$   
(opposite Svelto, Seigman)

- Now integral is a F.T. of a complex Gaussian=Gaussian

$$U(x,y,z) = \frac{i}{i\lambda z} e^{ik \frac{x^2+y^2}{2z}} \iint e^{+ik \frac{x'^2+y'^2}{2q}} e^{i2\pi(f_x x' + f_y y')} dx' dy'$$

$$\rightarrow u(r,z) = \frac{1}{q(z)} e^{-ik \frac{r^2}{2q(z)}} \quad \frac{1}{q(z)} = \frac{1}{R(z)} - i \frac{\lambda}{\pi w^2(z)}$$

## Complex q form for Gaussian beam

- This combines beam size and radius of curvature into one complex parameter
  - This form is used for ABCD calculations

$$A(r, z) = A_0 \frac{1}{1 + i\xi} e^{-\frac{r^2}{w_0^2(1+i\xi)}} \quad \rightarrow \quad A(r, z) = \frac{1}{q(z)} e^{-ik\frac{r^2}{2q(z)}}$$

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i \frac{\lambda}{\pi w^2(z)}$$

$$\frac{1}{q(z)} = \frac{1}{z + iz_R} = \frac{z}{z^2 + z_R^2} - i \frac{z_R}{z^2 + z_R^2}$$

$$= \frac{1}{R(z)} - i \frac{w_0^2}{z_R w^2(z)}$$

$$= \frac{1}{R(z)} - i \frac{\lambda}{\pi w^2(z)} = \frac{1}{R(z)} - i \frac{1}{Z(z)}$$

$$\frac{1}{R(z)} = \frac{1}{z \left(1 + \frac{z_R^2}{z^2}\right)} = \frac{z}{z^2 + z_R^2}$$

$$\frac{1}{w^2(z)} = \frac{1}{w_0^2 \left(1 + \frac{z^2}{z_R^2}\right)} = \frac{z_R^2}{w_0^2 (z^2 + z_R^2)}$$

## Complex q into standard form

$$u(r, z) = \frac{1}{q(z)} e^{-ik\frac{r^2}{2q(z)}} \quad \text{with} \quad \frac{1}{q(z)} = \frac{1}{R(z)} - i \frac{\lambda}{\pi w^2(z)}$$

Expand exponential:

$$\exp\left[-ik\frac{r^2}{2q(z)}\right] = \exp\left[-ik\frac{r^2}{2} \left(\frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}\right)\right]$$

$$= \exp\left[-ik\frac{r^2}{2} \frac{1}{R(z)} - i\frac{2\pi}{\lambda} \frac{r^2}{2} \left(-i\frac{\lambda}{\pi w^2(z)}\right)\right] = e^{-ik\frac{r^2}{2R(z)}} e^{-\frac{r^2}{w^2(z)}}$$

$$a + ib = \sqrt{a^2 + b^2} e^{i\arctan(b/a)}$$

Expand leading inverse q:

$$\frac{1}{q(z)} = -i \left( \frac{iz}{z^2 + z_R^2} + \frac{z_R}{z^2 + z_R^2} \right) = -i \left( \frac{\sqrt{z^2 + z_R^2}}{z^2 + z_R^2} \right) e^{i\arctan(z/z_R)}$$

$$= -i \left( \frac{1}{z_R \sqrt{1 + z^2/z_R^2}} \right) e^{i\arctan(z/z_R)} = \frac{w_0}{iz_R w(z)} e^{i\eta(z)}$$

### Standard form of Gaussian beam solutions

$$E(r, z, t) = A_0 e^{-i(kz - \omega t)} \frac{w_0}{w(z)} e^{-\frac{r^2}{w^2(z)}} e^{-i\frac{kr^2}{2R(z)}} e^{i\eta(z)}$$

Beam maintains a Gaussian profile as it propagates

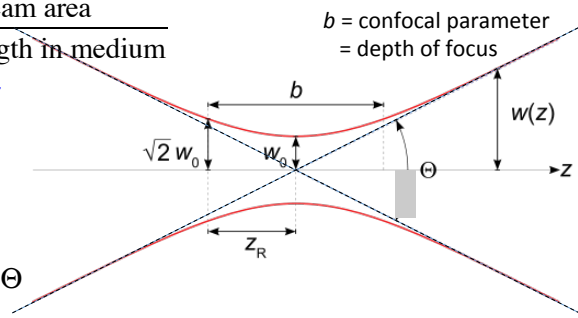
- beam radius that varies with z
- Origin of z coordinate is at the beam waist
- Rayleigh length  $z_R$  defines collimation distance from focal plane

$$z_R = \frac{\pi w_0^2}{\lambda / n} = \frac{\text{beam area}}{\text{wavelength in medium}}$$

$$w(z) = w_0 \sqrt{1 + \frac{z^2}{z_R^2}}$$

Geometric limit for  $z \gg z_R$

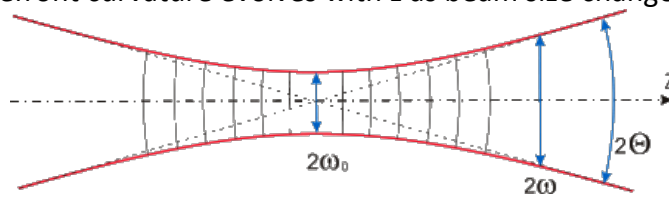
$$w(z) = z \frac{w_0}{z_R} = z \tan \Theta$$



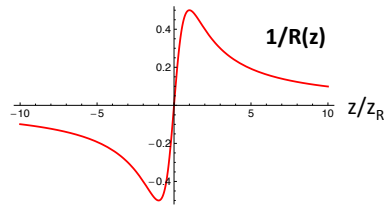
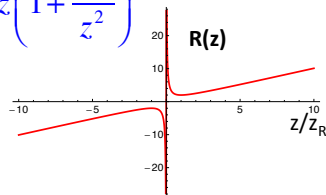
### Evolution of wavefronts

$$E(r, z, t) = A_0 e^{-i(kz - \omega t)} \frac{w_0}{w(z)} e^{-\frac{r^2}{w^2(z)}} e^{-i\frac{kr^2}{2R(z)}} e^{i\eta(z)}$$

- Wavefront curvature evolves with z as beam size changes



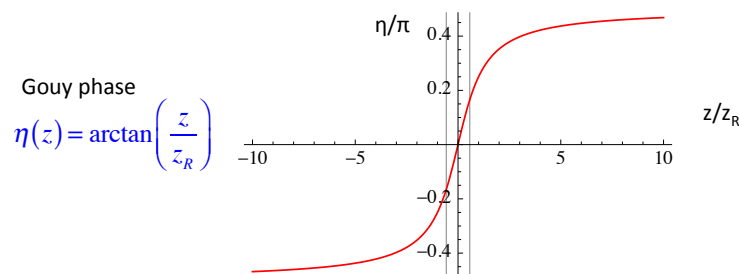
$$R(z) = z \left( 1 + \frac{z_R^2}{z^2} \right)$$



### On-axis phase: Gouy phase

$$E(r, z, t) = A_0 e^{-i(kz - \eta(z) - \omega t)} \frac{W_0}{w(z)} e^{-\frac{r^2}{w^2(z)}} e^{-i \frac{kr^2}{2R(z)}}$$

- Because the wavefront changes from focusing to defocusing, on-axis phase advances with z



### Difference between Siegman's complex q and standard form

$$u(r, z) = \frac{1}{q(z)} e^{-ik \frac{r^2}{2q(z)}} = \frac{1}{iz_R} \frac{W_0}{w(z)} e^{i\eta(z)} e^{-ik \frac{r^2}{2R(z)}} e^{-\frac{r^2}{w^2(z)}}$$

$$E(r, z, t) = A_0 \frac{W_0}{w(z)} e^{i(kz - \omega t)} e^{-\frac{r^2}{w^2(z)}} e^{i \frac{kr^2}{2R(z)}} e^{-i\eta(z)}$$

- Siegman's form for the complex q is used almost everywhere for the ABCD calculations.
- He uses the exp[+ iωt] convention, which accounts for the sign difference in the complex exponentials.
- With exp[-iωt] convention, define q as:

$$\frac{1}{q(z)} = \frac{1}{R(z)} + i \frac{\lambda}{\pi w^2(z)} = \frac{1}{z - iz_R}$$

### Compare Boyd's form to standard form

- Boyd's complex form is consistent with standard Gaussian beam form

$$A(r, z) = A_0 \frac{1}{1+i\xi} e^{-\frac{r^2}{w_0^2(1+i\xi)}} = A_0 \frac{1}{1+iz/z_R} e^{-\frac{r^2}{w_0^2(1+iz/z_R)}} \quad \xi \equiv \frac{z}{z_R}$$

$$\frac{1}{1+i\xi} = \frac{1}{1+iz/z_R} = \frac{z_R}{z_R+iz} = \frac{-iz_R}{z-iz_R} = \frac{-iz_R}{q(z)}$$

$$A(r, z) = A_0 (-iz_R) \frac{1}{q(z)} e^{+\frac{iz_R r^2}{w_0^2 q(z)}} = -iz_R A_0 \frac{1}{q(z)} e^{+\frac{ikr^2}{2q(z)}}$$