MATH 235 - SPRING 2008

HOMEWORK 2

1 (a) SHOW THAT $y(t)=e^{2t}+Ce^{t}$ $C \in \mathbb{R}$

IS A SOLUTION TO $\frac{dy}{dt}-y=e^{2t}$

$y=e^{2t}+Ce^{t}$

$\frac{dy}{dt}=2e^{2t}+Ce^{t}$

SUBSTITUTING INTO THE O.D.E. WE GET

$2e^{2t}+Ce^{t}-(e^{2t}+Ce^{t})=e^{2t}$

$\Rightarrow \quad e^{2t}=e^{2t}$

(b) SHOW THAT $x^2+y^2=Cx$ $C \in \mathbb{R}$

IS A SOLUTION TO $2xy\frac{dy}{dx}=y^2-x^2$

$\frac{d}{dx}\left[x^2+y^2=Cx\right] \Rightarrow 2x+2y\frac{dy}{dx}=C \Rightarrow \frac{dy}{dx}=\frac{C-2x}{2y}$

SUBSTITUTING INTO THE O.D.E.:

$2xy\left(\frac{C-2x}{2y}\right)=y^2-x^2 \Rightarrow Cx-2x^2=y^2-x^2$

$\Rightarrow (x^2+y^2)-2x^2=y^2-x^2 \Rightarrow y^2-x^2=y^2-x^2$

(c) SHOW THAT $y(t)=C_1\sinh(t)+C_2\cosh(t)$

IS A SOLUTION TO $y''-y=0$

$y=C_1\sinh(t)+C_2\cos(t)$

$y'=C_1\cosh(t)+C_2\sinh(t)$

$y''=C_1\sinh(t)+C_2\cosh(t)$

SUBSTITUTING INTO THE O.D.E.:

$C_1\sinh(t)+C_2\cosh(t)-(C_1\sinh(t)+C_2\cosh(t))=0$

$\Rightarrow 0=0$

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(d) SHOW THAT $y(t) = C_1\cos(t) + C_2\sin(t)$ $C_1, C_2 \in \mathbb{R}$ IS A SOLUTION TO $y'' + y = 0$

$$y = C_1\cos(t) + C_2\sin(t)$$

$$y' = -C_1\sin(t) + C_2\cos(t)$$

$$y'' = -C_1\cos(t) - C_2\sin(t)$$

SUBSTITUTING INTO THE O.D.E. :

$$-C_1\cos(t) - C_2\sin(t) + C_1\cos(t) + C_2\sin(t) = 0$$

$$\Rightarrow 0 = 0$$

(e) SHOW THAT $x(t) = Ae^{-Kt}$ AND

$$y(t) = \frac{K_1 A}{K_1 - K_2} e^{-K_2 t} + \frac{K_1 A}{K_2 - K_1} e^{K_1 t}$$

ARE SOLUTIONS TO THE SYSTEM OF O.D.E.'S

$$\frac{dx}{dt} = -K_1 x \qquad x(0) = A$$

$$\frac{dy}{dt} = K_1 x - K_2 y \qquad y(0) = 0$$

$$x = Ae^{-K_1 t}$$

$$\frac{dx}{dt} = -K_1 A e^{-K_1 t} = -K_1 x$$

$$y = \frac{K_1 A}{K_1 - K_2} e^{-K_2 t} + \frac{K_1 A}{K_2 - K_1} e^{-K_1 t}$$

$$\frac{dy}{dt} = \frac{-K_2 K_1 A e^{-K_2 t}}{K_1 - K_2} + \frac{-K_1^2 A e^{-K_1 t}}{K_2 - K_1}$$

$$= \frac{-K_2 K_1 A e^{-K_2 t}}{K_1 - K_2} + \frac{-K_1^2 A e^{-K_1 t} + K_1 K_2 A e^{-K_1 t} - K_1 K_2 A e^{-K_1 t}}{K_2 - K_1}$$

TOPS. 35500

HW # 1 - Solutions MACS325

1. a. Assumptions:

- The rate of growth is negative if the population is too small.
- The rate of growth is negative if the population is too big.
- If $P=0$ then $\frac{dP}{dt}=0$.

Variables/Parameters

- t - time, (independent)
- P - Population (dependent)
- k - growth rate (parameter)
- N - Carrying Capacity (Parameter)
- M - Sparsity const (Parameter)

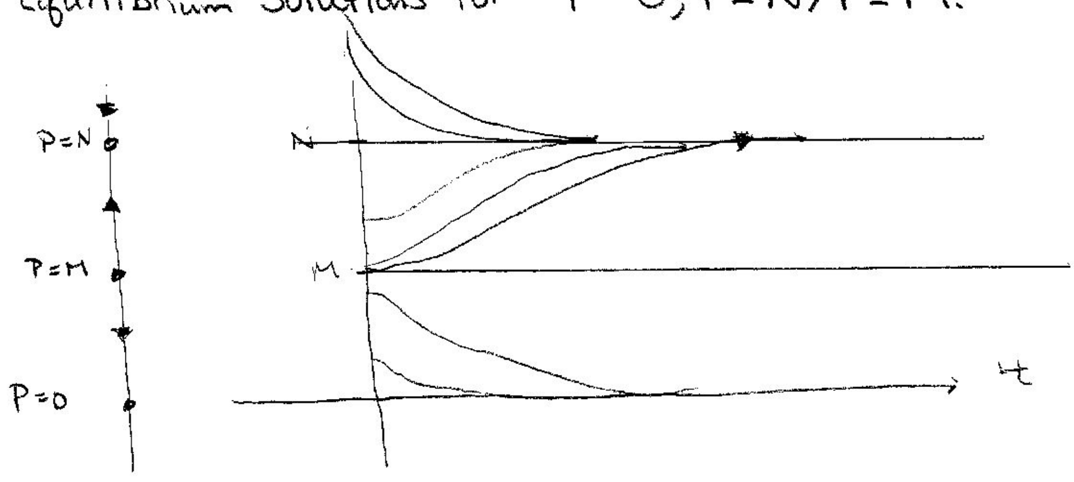
Model:

$$\frac{dP}{dt} = kP(1-P/N)\left(\frac{P}{M}-1\right), \text{ where } M<N.$$

Note: if $0<P<M \Rightarrow (P/M - 1)<0 \Rightarrow \frac{dP}{dt}<0$

$P>N \Rightarrow (1-P/N)<0 \Rightarrow \frac{dP}{dt}<0$

b. Equilibrium Solutions for $P=0$, $P=N$, $P=M$.



c. $\frac{dp}{dt} = kP(1-P/N)(P/M-1) \Leftrightarrow \frac{dp}{kP(1-P/N)(P/M-1)} = dt$

$C(kP)(1-P/N)$

$\Rightarrow$ (Partial Fractions)

$$\frac{1}{kP(1-P/N)(P/M-1)} = \frac{A}{kP} + \frac{B}{(1-P/N)} + \frac{C}{(P/M-1)} = \frac{A(1-P/N)(P/M-1) + B(kP)(P/M-1)}{kP(1-P/N)(P/M-1)}$$

$$\Rightarrow A(1-P/N)(P/M-1) + B(kP)(P/M-1) + C(kP)(1-P/N) = 1$$

for $P=0$ $A=-1$

$P=N$ $B(kN)(N/M-1) = 1 \Rightarrow B = \frac{1}{kN(N/M-1)}$

$P=M$ $C = \frac{1}{kM(1-M/N)}$

Thus

$$\int \frac{dp}{kP(1-P/N)(P/M-1)} = \int \left\{ \frac{-1}{kP} + \frac{[(kN)(N/M-1)]^{-1}}{1-P/N} + \frac{[kM(1-M/N)]^{-1}}{P/M-1} \right\} dp =$$

$$= -\ln|kP| - N[kN(N/M-1)]^{-1}\ln|1-P/N| + M[kM(1-M/N)]^{-1}\ln|P/M-1| =$$

$$= \ln\left|\frac{(P/M-1)^{M[kM(1-M/N)]^{-1}}}{kP(1-P/N)^{N[kN(N/M-1)]^{-1}}}\right| = \ln\left|\frac{(P/M-1)^{\frac{1}{k(1-M/N)}}}{kP(1-P/N)^{\frac{1}{k(N/M-1)}}}\right| = \int dt =$$

$$= t + C$$

Thus,

$$\frac{\left(\frac{P}{M}-1\right)^{\frac{1}{k(1-M/N)}}}{kP\left(1-P/N\right)^{\frac{1}{k(N/M-1)}}} = \pm e^{c}e^{t} = Ke^{t}, \quad K \in \mathbb{R}.$$

2. a) In this case

$x(t)$ is the prey

$y(t)$ is the predator.

Why?

- Terms with both x, y are interaction terms and dictate the effect x, y have on Each other.

Since (1) has a negative $y\sqrt{x}$ term the Effect of the interaction on the growth of $x(t)$ is negative. Thus, $x(t)$ decays if $y(t)$ is large. AKA y Eats x.

Similarly $y(t)$ is the prey since the interaction is favorable

b. If $x(t)=0$ for all t then $\frac{dy}{dt}=0 \Rightarrow y(t)=C$, $C\in\mathbb{R}$.

There must be a secondary food source.

a. $\frac{dy}{dt} = 1 + \frac{1}{y^2} \iff \frac{dy}{1 + 1/y^2} = \int \frac{y^2}{1+y^2} dy = \int dt \iff$

$\iff \int \frac{y^2}{1+y^2} dy = \int \left\{ 1 - \frac{1}{1+y^2} \right\} dy = y - \arctan(y) = t + c$

$\Rightarrow \quad y - \arctan(y) = t + c \qquad$ (Implicit)

b. $(y')^2 - xy' + y = (y')^2 - xy' + \frac{x^2}{4} - \frac{x^2}{4} + y =$

$= \left(y' - \frac{x}{2} \right)^2 + y - \frac{x^2}{4} = 0 \qquad (1)$

Let $z = -y + \frac{x^2}{4} \Rightarrow \frac{dz}{dx} = -\frac{dy}{dx} + \frac{2x}{4} = -y' + \frac{x}{2} = z'$

thus (1) becoms

$\left(y' - \frac{x}{2} \right)^2 + y - \frac{x^2}{4} = (-z')^2 - z = 0 \iff$

$\iff \quad \frac{dz}{dx} = -\sqrt{z} \iff \int \frac{dz}{\sqrt{z}} = \int -dx$

$\Rightarrow$

$2\sqrt{z} = -x + c \iff \sqrt{z} = -\frac{x}{2} + c \quad , \quad -\frac{x}{2} + c > 0$

$\Rightarrow \quad z = \left(-\frac{x}{2} + c \right)^2 = -y + \frac{x^2}{4} \Rightarrow y(x) = \frac{x^2}{4} - \left(-\frac{x}{2} + c \right)^2 =$

$= -cx + c^2 \; , \; -\frac{x}{2} + c > 0.$

c $\frac{dy}{dt} = (y^2+1)t$, $y(0)=1$

$\Rightarrow \int \frac{dy}{y^2+1} = \int t\,dt \iff \arctan(y) = \frac{1}{2}t^2 + c$

$\Rightarrow y(t) = \tan\left(\frac{1}{2}t^2 + c\right)$ [general soln]

Application of the initial condition:

$y(0) = 1 = \tan(0+c) \Rightarrow c = \frac{\pi}{4}$ (any $n\pi$ multiple should work as well)

$\Rightarrow$

$y(t) = \tan(\frac{1}{2}t^2 + n\frac{\pi}{4})$ [is the unique soln to the IVP.]

4. Assume that $y(t) = 2$, $\forall t \in \mathbb{R}$ is a solution to $\frac{dy}{dt} = f(t,y)$.

a) If $y(t)=2$ then $\frac{dy}{dt} = 0 = f(t,2)$, $\forall t \in \mathbb{R}$

b) From this information all we can sketch is the slope field for $y=2$.

c) From this we know that solutions starting above $y=2$ must stay above $y=2$ and soln below $y=2$ must stay below $y=2$ for all time.

5. a) See attached

b) See attached.

c) The Roots of $p(y)$ are Equilibrium solutions to the differential Equation $\frac{dy}{dt} = p(y)$.

d) See attached.

1.4 Example 1 - B

k	t_k	y_k	f(t_k,y_k)	delta t
0	0	0.5	0.875	0.125
1	0.125	0.609375	0.554966	
2	0.25	0.67874575	0.329813	
3	0.375	0.7199724	0.18685	
4	0.5	0.74332866	0.102626	
5	0.625	0.75615688	0.055336	
6	0.75	0.76307388	0.029528	
7	0.875	0.76676491	0.015667	
8	1	0.76872332	0.008287	
9	1.125	0.76975925	0.004377	
10	1.25	0.77030632	0.002309	
11	1.375	0.77059498	0.001218	
12	1.5	0.77074722	0.000642	
13	1.625	0.7708275	0.000339	
14	1.75	0.77086982	0.000178	
15	1.875	0.77089213	9.41E-05	
16	2	0.77090389	4.96E-05	
17	2.125	0.77091009	2.61E-05	
18	2.25	0.77091335	1.38E-05	
19	2.375	0.77091508	7.26E-06	
20	2.5	0.77091598	3.83E-06	
21	2.625	0.77091646	2.02E-06	
22	2.75	0.77091672	1.06E-06