Diffraction Theory
(Integral equations, Green's Functions)

Diffraction theory:

single slit

How to deal with this?

What you typically do is assume that at the ~~s~~creen, the field is zero away from the slit and some E_0 in the slit.

Given that, we can use Green's Fcns,

If you have some 2^{nd} order linear differential equation that looks like

$$L f(\vec{r},t) = \underbrace{g(\vec{r},t)}_{\text{known forcing function.}}$$

Example mass on a spring

$$\sum F_y = -mg + k(y-y_0) + \vec{F}_z(t) = m\frac{d^2y}{dt^2}$$

$$m\frac{d^2y}{dt^2} - k(y-y_0) + mg = F_z(t)$$

$$\underbrace{\left(m\frac{d^2}{dt^2} - k\right)}_{L} \underset{\uparrow\, f}{y} = \underbrace{F_z(t) - ky_0 - mg}_{g}$$

If you find green's function, that means you solve

$$\left(m\frac{d^2}{dt^2} - k\right) G = -\delta(t-t')$$

$\Rightarrow$ you can find $y(t) = -\int_{-\infty}^{t} G(t-t')\,g(t')\,dt'$

$$+\,y\big|_{-\infty}$$

For EM waves:

$$\nabla^2 \vec{E} + k^2 \vec{E} = \emptyset$$

Free space.

$$L = (\nabla^2 + k^2)$$

For t-dependent problems:

$$L = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)$$

Reminder: $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) V = \frac{-\rho}{\epsilon_0}$

$$V = \int \frac{1}{4\pi\epsilon_0} \frac{\rho(\vec{r}\,', t_r)}{\mathscr{r}} d\tau'$$

$$\Rightarrow G(\vec{r}, \vec{r}\,', t, t_r) = \frac{1}{4\pi \mathscr{r}}$$

If you have harmonic t-dep:

$$\rho(\vec{r}\,', t_r) = \mathrm{Re}\left(\tilde{\rho}(\vec{r}\,') e^{-i\omega t_r}\right)$$

$$= \rho(\vec{r}\,') \cos(\omega t_r + \delta)$$

$$\Rightarrow V(\vec{r}, t) = \tilde{V}(\vec{r}) e^{-i\omega t} = \frac{1}{4\pi\epsilon_0} \int \frac{\tilde{\rho}(\vec{r}\,') e^{-i\omega t_r}}{\mathscr{r}} d\tau'$$

$$t = t_r + \frac{\mathscr{r}}{c} \Rightarrow t_r = t - \frac{\mathscr{r}}{c}$$

$$= \frac{1}{4\pi\epsilon_0} \int \frac{\tilde{\rho}(\vec{r}\,') e^{-i\omega t} e^{+i\omega/c\, \mathscr{r}}}{\mathscr{r}} d\tau'$$

$$\tilde{V}(\vec{r}) = \int \frac{\tilde{\rho}(\vec{r}\,')}{\epsilon_0} \cdot \frac{e^{i\omega/c\, \mathscr{r}}}{4\pi \mathscr{r}} d\tau'$$

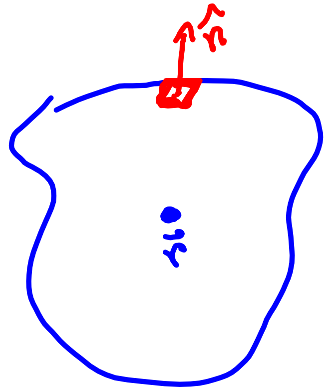
$$G = \frac{e^{i\frac{\omega}{c}\mathscr{r}}}{4\pi\mathscr{r}} = \frac{e^{ik\mathscr{r}}}{4\pi\mathscr{r}}$$

$G = \frac{e^{ik\mathscr{r}}}{4\pi\mathscr{r}}$ is the Green's fn for $L = (\nabla^2 + k^2)$

It solves $(\nabla^2 + k^2) G = -\delta(\vec{r} - \vec{r}\,')$

There's another way

It turns out that for a closed surface with no charges or currents inside, that

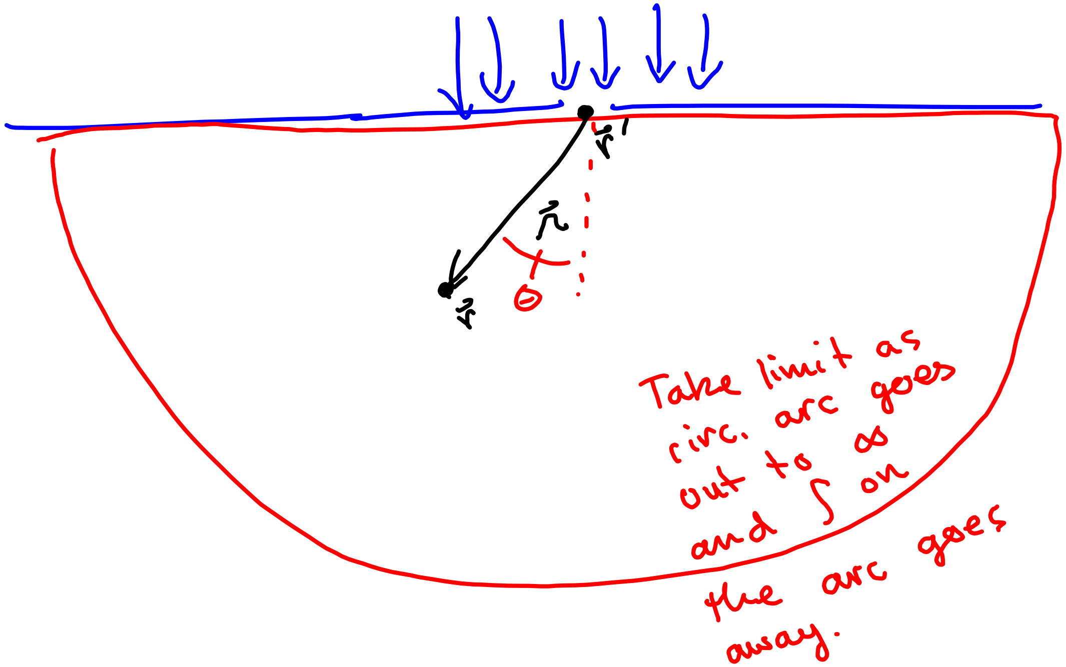


$$\vec{E}(\vec{r}) = \oint_S \vec{E}\left(\frac{\partial G}{\partial n'}\right) - G\left(\frac{\partial \vec{E}}{\partial n'}\right) ds'$$

$$(\vec{\nabla} G \cdot \hat{n}) \qquad (\vec{\nabla}\vec{E})\cdot\hat{n}$$

$$E_y(\vec{r}) = \oint_S \vec{E}(\vec{r}\,')\,\frac{\partial G(\vec{r},\vec{r}\,')}{\partial n'} - G(\vec{r},\vec{r}\,')\,\frac{\partial E(\vec{r}\,')}{\partial n'}\, ds'$$

Our original problem



You can rewrite this a lot of ways. The most ubiquitous is the Kirkhoff Int.

$$\vec{E}(\vec{r}) = \frac{-i}{\lambda} \int \frac{e^{ik\pi}}{\pi} \cos\theta \, E(\vec{r}') \, ds'$$

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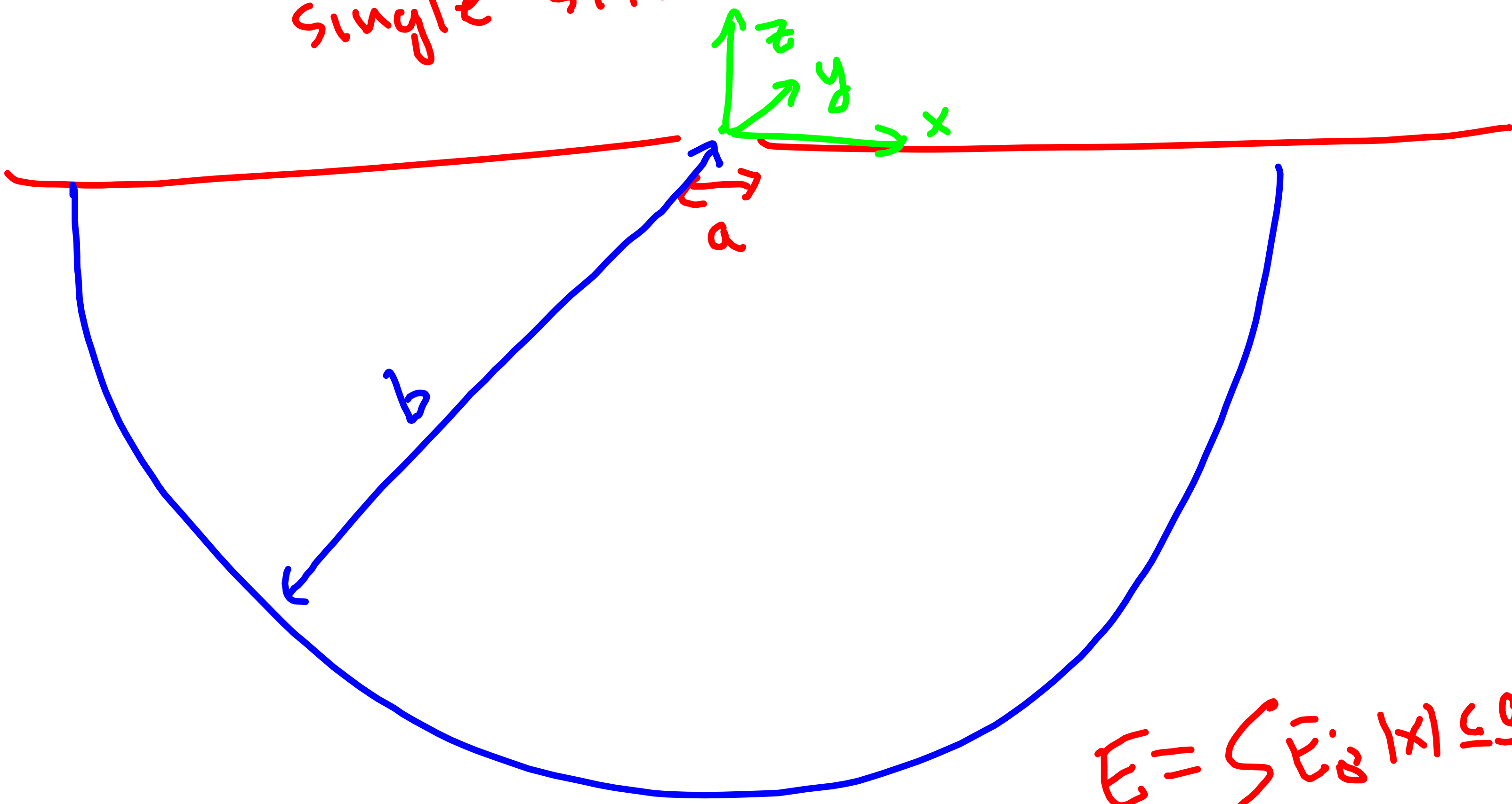
Let's do that for the single slit.



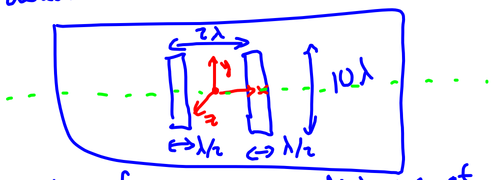
$$E = \begin{cases} E_0 \, ; & |x| \le \frac{a}{2} \\ 0 \, ; & |x| > \frac{a}{2} \end{cases}$$

Extra Credit

1) 5% on a test:
Find my mistake in last lecture

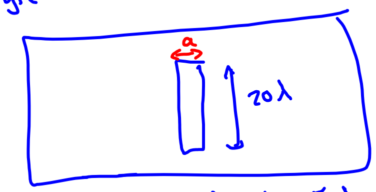
2) Set up a mathematica notebook that can solve for $\vec{E}$ below any screen (it takes an arbitrary $\vec{E}(x', y')$ as an input).
Let's do one polarization so $\vec{E}(x', y') = E_x(x', y')$

a) double slit



Solve for E_x at a distance of $\lambda, 5\lambda, 10, 20\lambda$ from screen at $y = \emptyset$.

b) single slit



Vary a: 0.1λ, 0.5λ, λ, 5λ, 10λ, 20λ

Plot E_x at a distance of 20λ from screen at $y = \emptyset$.

Also worth 5% of test.