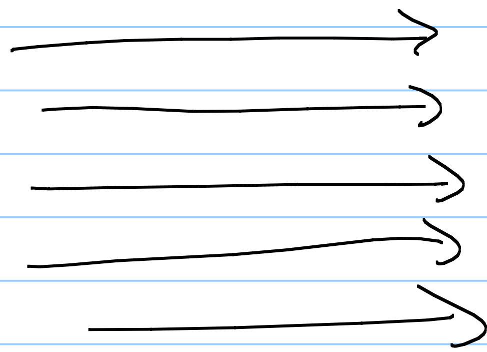
Note Title 12_3_07 12/3/2007

Example



infinite plane parallel electric field in z direction

grounded cond. sphere placed in E field

find E field outside sphere

first we'll find V (potential) then $E = -\nabla V$

Solutions must be symmetric about z axis : no ϕ dep.

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$$

BC 1 : $V(r=a, \theta) = 0$ (grounded)

$$0 = \sum_l \left(A_l a^l + B_l a^{-(l+1)}\right) P_l(\cos\theta)$$

The P_ℓ are orthogonal

$$\int_{-1}^{1} P_\ell(x)\, P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

So multiply both sides by $P_{\ell'}$ and integrate

$$0 = \sum_{\ell=0}^{\infty}\left(A_\ell a^\ell + B_\ell a^{-(\ell+1)}\right)\underbrace{\int_{-1}^{1} P_\ell(x)\, P_{\ell'}(x)\, dx}_{\frac{2}{2\ell+1}\delta_{\ell\ell'}}$$

$$= A_\ell a^\ell + B_\ell a^{-(\ell+1)} = 0$$

$$\Rightarrow \boxed{B_\ell = -a^{2\ell+1} A_\ell}$$

BC 2: as $r \to \infty$ $\vec{E} \Rightarrow E_0\hat{z}$

this means that $V(r,\theta) \to -E_0 z$

Look at $V(r,\theta) = \sum\left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right)P_\ell$
as $r \to \infty$ the B_ℓ terms drop out of the sum so

also notice that $z = r\cos\theta$ so

$$\lim_{r\to\infty} V(r,\theta) = -E_0 z = -E_0 r\cos\theta$$

$$= -E_0 r P_1(\cos\theta)$$

$$= \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

clearly all coeff. are zero except $l=1$

$$A_1 r P_1(\cos\theta) = -E_0 r P_1(\cos\theta)$$

$$\Rightarrow \quad A_1 = -E_0$$

And from $\boxed{B_l = -a^{2l+1} A_l}$

it follows that

$$B_1 = -a^3 A_1$$

$$\Rightarrow V(r,\theta) = \left(-E_0 r + E_0 \frac{a^3}{r^2}\right)\cos\theta$$

$$= -E_0\left(1 - \left(\frac{a}{r}\right)^3\right) r\cos\theta$$

Finally take gradient to get E_r & E_θ.