MATH348: VIBRATIONAL MODES OF A IDEALLY THIN RECTANGULAR MEMBRANE

The sea is rollin' like a thousand-pound keg. We're goin' surfin', goin' surfin'.

Abstract. Recall that for a wave on an ideal string with fixed ends we found,

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\sqrt{\lambda_n} x\right)\left[A_n \cos\left(c\sqrt{\lambda_n} t\right) + \sin\left(c\sqrt{\lambda_n} t\right)\right], \tag{1}$$

whose spatial solutions were derived from the boundary-value problem,

$$X'' + \lambda X = 0, \quad x \in (0, L) \tag{2}$$

$$X(0) = 0, \quad X(L) = 0. \tag{3}$$

This will be the building block for studying the problem in multiple dimensions of space. We begin with a short introduction and then proceed to what are essential notes about the problem of vibrations of a thin rectangular membrane. After this we conclude with action items for a student to do to fill out the details of the problem.

1. Introduction

It is possible to generalize the derivation of time dynamics of an ideally elastic medium to any finite number of spatial dimensions. Doing so one arrives at

$$u_{tt} = c^2 \triangle u \tag{4}$$

where $\triangle$ is known as the Laplacian and, when applied to a function, sums the second spatial derivatives of the unknown function, $\triangle u = \sum_{i=1}^{n} u_{x_i x_i}$. If we assume that space can be separated from time, $u(\mathbf{x}, t) = T(t)F(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ then you get the following equations

$$T'' + c^2 \lambda T = 0, \tag{5}$$

$$\triangle F + \lambda F = 0. \tag{6}$$

The second of these two equations is another PDE known as Helmholtz's. It is possible to continue to apply separation of variables to Helmholtz equation. The following studies this in the context of a rectangular sub-domain of 2-space, $D \subset \mathbb{R}^2$.

2. Modes of a Two-Dimensional Ideal Membrane

We begin with

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad \begin{array}{c} x \in (0, L_x), \\ y \in (0, L_y), \\ t \in (0, \infty), \end{array} \tag{7}$$

$$u(0, y, t) = 0, \quad u(0, L_x, t) = 0, \quad u(x, 0, t) = 0, \quad u(x, L_y, t) = 0, \tag{8}$$

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which describes waves on a rectangle of area $A = L_x L_y$, whose edges are all fixed. We can think of this as a rectangular drum-head. We can solve this problem by application of separation of variables to the Helmholtz equation,

$$F_{xx} + F_{yy} + \lambda_1 F = 0. \tag{9}$$

Specifically, assuming $F(x,y) = X(x)Y(y)$ gives the following boundary-value problems,

$$X'' + \lambda_2 X = 0, \quad X(0) = 0,\ X(L_x) = 0, \tag{10}$$

$$Y'' + \lambda_3 Y = 0, \quad Y(0) = 0,\ Y(L_y) = 0, \tag{11}$$

where $\lambda_1 = \lambda_2 + \lambda_3$. Solving each of these gives and recombining the result separated solutions gives,

(12)

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin\left(\sqrt{\lambda_2(n)}x\right)\sin\left(\sqrt{\lambda_3(m)}y\right)\left[A_{nm}\cos\left(c\sqrt{\lambda_1(n,m)}t\right) + B_{nm}\sin\left(c\sqrt{\lambda_1(n,m)}\right)\right],$$

where

(13)

$$\lambda_1(n,m) = \lambda_2(n) + \lambda_3(m) = \frac{n^2\pi^2}{L_x^2} + \frac{m^2\pi^2}{L_y^2}, \quad n = 1,2,3,\ldots,\ m = 1,2,3,\ldots.$$

To understand this solution:

1. There is a Fourier series for each dimension of space. That is, if a Fourier sine series can create reasonable but fairly complicated one-dimensional graphs, then spatial wave-interference in two-dimensions should do the same.
2. To understand the *primitive/mode* dynamics we consider specific modes, $(n,m) = \{(1,1),(1,2),(2,1),(2,2)\}$

The $(1,2)-$mode takes the form,

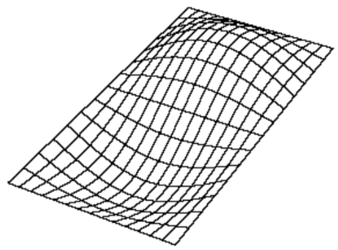
$$\begin{aligned} u_{1,2}(x,y,t) &= \sin\left(\sqrt{\lambda_2(1)}x\right)\sin\left(\sqrt{\lambda_3(2)}y\right)\left[A_{1,2}\cos\left(c\sqrt{\lambda_1(1,2)}t\right) + B_{1,2}\sin\left(c\sqrt{\lambda_1(1,2)}\right)\right] \\ &= \sin\left(\frac{\pi}{L_x}x\right)\sin\left(\frac{2\pi}{L_y}y\right)\left[A_{nm}\cos\left(c\sqrt{\lambda_1(n,m)}t\right) + B_{nm}\sin\left(c\sqrt{\lambda_1(n,m)}\right)\right] \\ &= \sin\left(\frac{\pi}{L_x}x\right)\sin\left(\frac{2\pi}{L_y}y\right)T_{1,2}(t). \end{aligned}$$

Notice that the time-dynamic is not really different than in the one-dimensional case. Namely, it is time-harmonic and depends on the spatial angular-frequencies. However, now it contains two frequencies, one for each spatial-wave. Anyhow, the spatial-waves are the interesting ones, specifically, their nodal structure. For this mode we notice:

1. In the $x-$direction, two lines that are fixed in time and that is the line defined by $x = 0$ and the line defined by $x = L$.
2. In the $y-$direction, there are three lines that are fixed in time. These lines are, $y = 0, y = L$ and $y = L/2$.

From this we can imagine, a rectangle cut into two strips, whose oscillate up and down relative to one another. For example,

The (1,2) Mode



To find the solution to the problem given the initial values, $u(x, y, 0) = f(x, y), u_t(x, y, 0) = g(x, y)$, we must be cleaver with our use of Fourier series and coefficients. Being so we can show that,

(14)
$$A_{nm} = \frac{4}{L_x L_y} \int_0^{L_x} \int_0^{L_y} f(x, y) \sin\left(\sqrt{\lambda_2(n)}x\right) \sin\left(\sqrt{\lambda_3(m)}y\right) dxdy$$

(15)
$$B_{nm} = \frac{4}{c\sqrt{\lambda_1(n, m)}L_x L_y} \int_0^{L_x} \int_0^{L_y} g(x, y) \sin\left(\sqrt{\lambda_2(n)}x\right) \sin\left(\sqrt{\lambda_3(m)}y\right) dxdy$$

3. Things to Do

If you can work out the details of a two-dimensional in space problem, then one-dimensional in space problems seem transparent in comparison. The following list are action items for the above discussion.

1. Assuming $u(x, y, t) = T(t)F(x, y)$ derive Eq. (9) from Eq. (7).
2. Assume that $F(x, y) = X(x)Y(y)$ and derive Eqs.(10) – (11) from Eq. (9).
3. Using the two counting variables $m, n \in \mathbb{N}^+$, solve Eqs. (10) – (11).
4. From your solutions form the general solution Eq. (12) and the define the first separation-constant Eq. (13).
5. Analyze the following modes $(n, m) = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.
6. Extra Credit [Notate this in your log]: Derive the unknown constants A_{nm}, B_{nm}, Eq. (14)–(15), by using your knowledge of Fourier coefficients.

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