

10-3-07

①

Note Title

10/3/2007

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}^T = [1 \ 2]$$

$$\begin{bmatrix} i & 2 \\ 3 & -i \end{bmatrix}^+ = \overline{\begin{bmatrix} i & 3 \\ 2 & -i \end{bmatrix}} = \begin{bmatrix} -i & 3 \\ 2 & i \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2} \\ i \end{bmatrix}^+ = \overline{\begin{bmatrix} \sqrt{2} & i \end{bmatrix}} = \begin{bmatrix} \sqrt{2} & -i \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2} \\ i \end{bmatrix}^+ \begin{bmatrix} \sqrt{2} \\ i \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -i \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ i \end{bmatrix} = 3$$

②

Real

Complex

$$A^T$$

$$A^\dagger = \overline{A^T}$$

Symmetric

Hermitian

$$A = A^T$$

$$A^\dagger = A$$

Orthogonal

Unitary

$$A^T A = A A^T = I$$

$$A^\dagger A = A A^\dagger = I$$

Both symmetric & Hermitian matrices have real λ -values

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Note Title

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Let $\vec{r}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\vec{r}_2 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

be 2 Σ -vectors of

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ with } \lambda_1, \lambda_2$$

$$\begin{aligned} A\vec{r}_1 &= \lambda_1 \vec{r}_1 \\ A\vec{r}_2 &= \lambda_2 \vec{r}_2 \end{aligned}$$

$$\begin{aligned} \text{Now } \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} &= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 y_1 \\ \lambda_1 x_2 & \lambda_2 y_2 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow \\ \vec{r}_1 & \vec{r}_2 \end{matrix} &= \begin{bmatrix} \lambda_1 \vec{r}_1 & \lambda_2 \vec{r}_2 \end{bmatrix} \end{aligned}$$

④

on the other hand

$$\begin{matrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} & \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} & = & \begin{bmatrix} a_{11}x_1 + a_{12}x_2 & a_{11}y_1 + a_{12}y_2 \\ a_{21}x_1 + a_{22}x_2 & a_{21}y_1 + a_{22}y_2 \end{bmatrix} \\ A & \begin{matrix} \vec{r}_1 & \vec{r}_2 \end{matrix} & & \begin{matrix} \underbrace{\hspace{10em}}_{A \cdot \vec{r}_1} & \underbrace{\hspace{10em}}_{A \cdot \vec{r}_2} \end{matrix} \end{matrix}$$

thus

$$A \begin{bmatrix} \vec{r}_1 & \vec{r}_2 \end{bmatrix} = \begin{bmatrix} \vec{r}_1 & \vec{r}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

is completely equivalent to

$$A \vec{r}_1 = \lambda_1 \vec{r}_1$$

$$A \vec{r}_2 = \lambda_2 \vec{r}_2$$

But now we see that if

$C \equiv \begin{bmatrix} \vec{r}_1 & \vec{r}_2 \end{bmatrix}$ is invertible

Then with $D \equiv \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

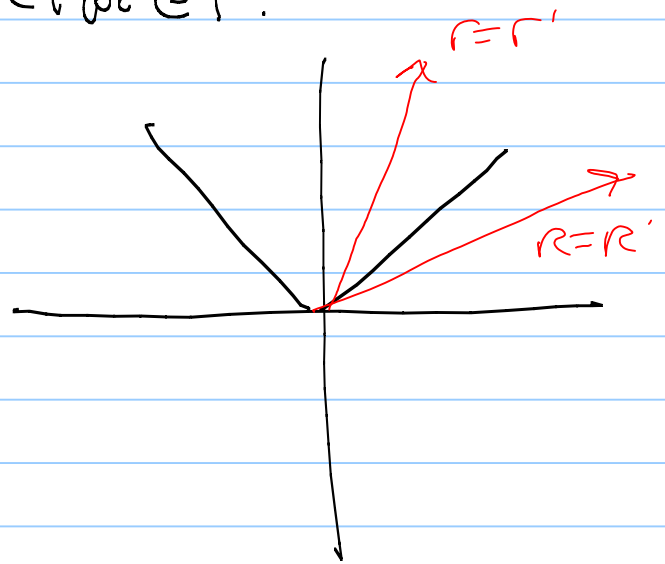
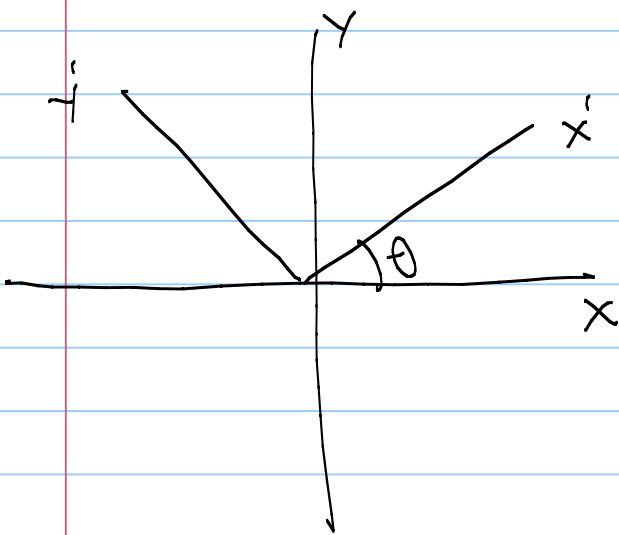
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$$AC = CD \Rightarrow$$

$$C^{-1}AC = D$$

↑
diagonal.

Geometrical Interpret.



rotation

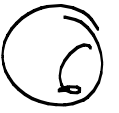
or

$$x = x' \cos \theta - y' \sin \theta$$

$$x' = x \cos \theta + y \sin \theta$$

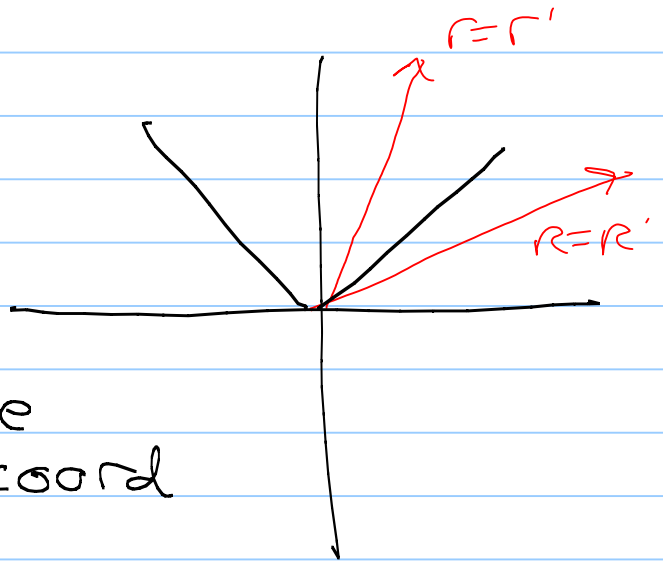
$$y = x' \sin \theta + y' \cos \theta$$

$$y' = -x \sin \theta + y \cos \theta$$



i.e. $\vec{r} = C \vec{r}'$

now \vec{r}, \vec{r}'
 \vec{r}, \vec{r}'



represent the same vector in 2 coord systems.

$$\vec{R} = C \vec{R}'$$

Consider the transf. in the unprimed coordinates that takes $\vec{R} \rightarrow \vec{R}'$
call it M

$$\vec{R} = M \vec{R}'$$

in the primed coordinates this becomes

$$C \vec{R}' = M C \vec{R}'$$

$$\Rightarrow \vec{R}' = C^{-1} M C \vec{R}'$$

⑦

So the transformation represented by M in the unprimed coordinates becomes $C^{-1} M C$ in the primed.

$$\vec{R} = M \vec{r}$$



$$\vec{R}' = C^{-1} M C \vec{r}'$$



Diagonalizing a matrix M amounts to transforming coordinates to one in which the transformation is diagonal. ⑧

Ex $A = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$

Char. polynomial : $\text{Det } |A - \lambda I| = 0$

$$\Rightarrow (5 - \lambda)^2 - 1 = 0$$

$$\Rightarrow 5 - \lambda = \pm 1$$

$$\Rightarrow \boxed{\lambda = 5 \pm 1}$$

now find Σ -vectors

(9)

$$\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 6 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{cases} 5x + y = 6x \\ x + 5y = 6y \end{cases} \quad \} \quad x=y$$

so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 4 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{cases} 5x + y = 4x \\ x + 5y = 4y \end{cases} \quad \} \quad x = -y$$

so $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\text{Let } Q = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right)$$

⑩

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{Let } \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} = M$$

we showed that

$$MQ = Q\Delta \quad \Delta = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$$

Q is orthogonal ($Q^T Q = I$)

$$\text{so } M = Q\Delta Q^T$$

lets
check

$$Q \Delta Q^T = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (11)$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 6 & 6 \\ 4 & -4 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 10 & 2 \\ 2 & 10 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \quad \checkmark$$

Any matrix with N distinct λ -values can be diagonalized

Any symmetric matrix can be diagonalized by an orthogonal matrix

Any Hermitian matrix can be diagonalized by a unitary matrix

(12)

what can go wrong?

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

char. poly:

$$(-\lambda)(-\lambda) = 0$$

$$\lambda^2 = 0$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow y = 0$$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an ξ -vector

for both ξ -values!

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

char polynomial

(B)

$$(3-\lambda)^2 = 0$$

$$3-\lambda = 0 \Rightarrow \begin{matrix} \lambda_1 = 3 \\ \lambda_2 = 3 \end{matrix}$$

repeated Σ -values

$$A \vec{r}_1 = 3 \vec{r}_1$$

$$A \vec{r}_2 = 3 \vec{r}_2$$

$$\vec{r}_1 \propto \vec{r}_2 \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

if N Σ -vectors correspond to N different Σ -values then

the Σ -vectors are linearly indep.

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$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

is symmetric

but any

invertible matrix
satisfies

$$C^{-1} M C = M$$

Since M is symmetric, we
are free to choose C
to be orthogonal.

Reminder

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A set of vectors is linearly independent, if none of them can be written as a linear comb. of finitely many other vectors in the set.

$$\begin{array}{c} \text{indep.} \\ \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}} \\ \text{depend.} \end{array}$$

$$9 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$